

Algebraic Schrödinger Representation of Quantum Chromodynamics in Temporal Gauge and Resolution of Constraints

H. Stumpf and W. Pfister

Institut für Theoretische Physik der Universität Tübingen, D-72076 Tübingen

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The algebraic formalism of QCD is expounded in order to demonstrate the resolution of Gauß constraints on the quantum level. In the algebraic approach energy eigenstates of QCD in temporal gauge are represented in an algebraic GNS-basis. The corresponding Hilbert space is mapped into a functional space of generating functional states. The image of the QCD-Heisenberg dynamics becomes a functional energy equation for these states. In the same manner the Gauß constraints are mapped into functional space. In functional space the Gauß constraints can be exactly resolved. The resolutions are defined by nonperturbative recurrence relations. The longitudinal color electric energy can be expressed by means of these resolvents, which leads to “dressed” color Coulomb forces in temporal gauge. Although present in the system, the longitudinal vector potentials do not affect its energy eigenvalues. This leads to a selfconsistent subsystem within the functional energy equation in temporal gauge which has to be identified with a functional energy equation in Coulomb gauge. In addition this procedure implies a clear conception for the incorporation of various algebraic representations into the formal Heisenberg dynamics and establishes the algebraic “Schrödinger” equation for QCD in functional space.

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1. Introduction

One of the outstanding and unsolved problems of quantum chromodynamics is the investigation of its low energy sector, although numerous papers about this topic have been published. In this sector mesons and nucleons are formed and this formation should be governed and explained by the effective forces arising from an appropriate evaluation of quantum chromodynamics. In gauge theories redundant variables occur in various gauges, and the derivation of effective forces in such theories is closely connected with the elimination of these redundant variables, as is demonstrated by the example of quantum electrodynamics. In order to maintain manifest covariance, covariant gauges cannot do without redundant variables and thus they are not suited for the evaluation of effective forces. Hence non-covariant gauges have to be applied. Non-covariant gauges were increasingly treated in the literature. For reviews see [1] - [4]. Although these gauges are not manifestly covariant it can be shown that they can be correctly Poincaré transformed by a mixture of gauge transformations

and Poincaré transformations [5] - [8]. In the perturbative range renormalizability is also secured [1], [9], [10].

Guided by the example of quantum electrodynamics the most popular approach consists in the elimination of redundant variables at the classical level. This is achieved by the introduction of gauge invariant variables, which are non-cartesian field “coordinates”, or equivalently by direct solution of the constraints [6], [11] - [19]. Apart from observations that this procedure is not consistent with direct canonical quantization [20] the preconditions for the success of this procedure are very restrictive already at the classical level [6], [13]. For canonical quantization a resolution of the constraints on the operator level was performed in axial gauge [21] - [23]. Frequently loop variables are proposed. But loop variables can only be exactly evaluated for quantum Maxwell fields [24]. In addition, inconsistencies have been discovered [25]. Apart from path integral manipulations which stem from the classical treatment and suffer from the same drawbacks, Schrödinger equations for the gauge fields are used [26] - [34]. But Schrödinger equations for fields suffer from the ill defined representation problem. Hence, with respect to these various drawbacks we

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do not follow any of these approaches in this paper. Rather, in treating this problem we try to establish an algebraic formalism for canonically quantized gauge fields, which is in accordance with the principle of algebraic representation theory. In order to avoid any difficulty with non-cartesian coordinates we use a strictly cartesian coordinate version of chromodynamics, which imperatively leads to the use of temporal gauge. The general algebraic formalism for quantum gauge fields was developed for the example of quantum electrodynamics in a foregoing paper [35] which we denote here by I. In the following we concentrate on the peculiarities of the nonabelian case.

In I we demonstrated that the evaluation of the Coulomb forces in quantum electrodynamics can be achieved without Coulomb gauge fixing if one starts with temporal gauge fixing conditions and residual gauge invariance constraints, and afterwards evaluates Coulomb forces by suitable transformations. In this way the Coulomb gauge is circumvented, but nevertheless the Coulomb forces are explicitly derived. In nonabelian theories temporal gauge fixing conditions can be exactly fulfilled [36]. It is the aim of this paper to show that also the invariance conditions against residual gauge transformation can be exactly resolved on the quantum level. As a consequence one then obtains explicit expressions for the effective “color” Coulomb forces in temporal gauge. In this way the “Coulomb” gauge is established on the quantum level, which on the classical level cannot be exactly calculated. This is a hint that the constraint problem in quantum theory is of another quality than in classical theory. In the literature so far this fact was failed to see, as neither by use of operator calculations nor by use of path integrals this difference between classical and quantum theories can be discovered. Both these approaches have too much affinities to the classical theory in leaving the representation problem out of consideration. In the present investigation the representation problem is taken into account by working with algebraic states or related quantities. In so doing we develop a method of treating the quantization of abelian and nonabelian gauge theories which, apart from the problem of existence for the whole theory, is completely exact, i.e., without any approximation.

With respect to a more detailed explanation and motivation for the algebraic method pursuit in this paper we refer to the introduction of the paper on quantum electrodynamics where these topics are extensively discussed.

2. Classical nonabelian gauge theory

We proceed in complete analogy to QED. We adopt the metric $g_{\mu\lambda} = \text{diag}(1, -1, -1, -1)$ with $x^\mu = (t, \mathbf{r})$ and $A_a^\mu = (A_a^0, \mathbf{A}_a)$ in accordance with I. We take into account the coupling of the nonabelian gauge field to fermions. The Lagrangian of this system reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \quad (1)$$

with

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf^{abc}A_\mu^b A_\nu^c. \quad (2)$$

The canonical conjugate momenta are

$$\pi_k^a := \frac{\delta\mathcal{L}}{\delta\partial_0 A_a^k} = F_{0k}^a = -E_k^a, \quad k = 1, 2, 3, \quad (3)$$

$$\pi_0^a := \frac{\delta\mathcal{L}}{\delta\partial_0 A_a^0} = 0. \quad (4)$$

We formulate the field equations resulting from the Lagrangian formalism in terms of canonically conjugate variables. In this form the classical field equations are

$$\begin{aligned} \partial_t \mathbf{E}_a &= \nabla \times \mathbf{B}_a + gf_{abc}(A_b^0 \mathbf{E}_c + \mathbf{A}_b \times \mathbf{B}_c) - \mathbf{j}_a, \\ \partial_t \mathbf{A}_a &= -\mathbf{E}_a - \nabla A_a^0 - gf_{abc} \mathbf{A}_b A_c^0. \end{aligned} \quad (5)$$

The constraints are

$$\begin{aligned} \nabla \cdot \mathbf{E}_a + gf_{abc} \mathbf{A}_b \cdot \mathbf{E}_c &= j_a^0, \\ \mathbf{B}_a &= \nabla \times \mathbf{A}_a + \frac{1}{2}gf_{abc}(\mathbf{A}_b \times \mathbf{A}_c) \end{aligned} \quad (6)$$

with

$$j_a^\nu := (\bar{\psi}\gamma^\nu L_a \psi)g, \quad (7)$$

and the fermion equations are

$$\begin{aligned} (i\gamma^\mu \partial_\mu + i\gamma^\mu igA_a^\mu L_a - m)\psi &= 0, \\ i(\partial_\mu + igA_a^\mu L_a)\bar{\psi}\gamma^\mu + m\bar{\psi} &= 0. \end{aligned} \quad (8)$$

The notation is standard and can be found elsewhere [36].

In temporal gauge $A_a^0 \equiv 0$ and with use of super-spinors these equations go over into

$$\begin{aligned}\dot{\mathbf{E}}_a &= \nabla \times \mathbf{B}_a + g f_{abc} (\mathbf{A}_b \times \mathbf{B}_c) - \mathbf{j}_a, \\ \dot{\mathbf{A}}_a &= -\mathbf{E}_a, \\ i\dot{\psi} &= -(i\alpha^k \partial_k - \beta m)\psi - g\gamma^0 \gamma^k A_k^a L_a \sigma^3 \psi.\end{aligned}\quad (9)$$

If we observe that $F_a^{ik} = -\epsilon^{ikj} B_a^j$, equations (9) can be rewritten as

$$\begin{aligned}\dot{A}_a &= -E_a, \\ \dot{E}_a^i &= -(\partial_k \delta_{ac} + g f_{abc} A_b^k) F_c^{ki} - j_a^i \\ &= (\partial_k \delta_{ac} + g f_{abc} A_b^k) \\ &\quad \cdot (\partial_k A_c^i - \partial_i A_c^k + g f_{cb'c'} A_{b'}^k A_{c'}^i) - j_a^i.\end{aligned}\quad (11)$$

The Hamiltonian belonging to this system is given by

$$\begin{aligned}H &= \frac{1}{2} \int d^3r [E_k^a E_k^a + B_k^a B_k^a] \\ &\quad + \int d^3r \psi^\dagger \left(\frac{1}{i} \vec{\alpha} \cdot \mathbf{D} + m\beta \right) \psi\end{aligned}\quad (12)$$

if formulated in terms of conjugate variables. In this case H shows a dependence on A^0 for $A^0 \neq 0$, but it can be demonstrated that the $A^0 \equiv 0$ gauge is possible [36]. If one tries to derive the field equations by the canonical Hamilton formalism one obtains only equations (9), (10), but not the Gauß law (6). As in QED, this is caused by the fact that the Hamiltonian (12) still admits residual gauge transformations [36] with $\omega = \omega(\mathbf{r}) \in C_0^\infty(\mathbb{R}^3)$. These gauge transformations are given by

$$\begin{aligned}\frac{1}{2} L_a A_k'^a &= \frac{1}{2} U(\omega) L_a A_k^a U^{-1}(\omega) \\ &\quad - \frac{i}{g} [\partial_k U(\omega)] U^{-1}(\omega),\end{aligned}\quad (13)$$

$$L_a E_k'^a = U(\omega) L_a E_k^a U^{-1}(\omega), \quad (14)$$

$$\begin{aligned}\psi'(\mathbf{r}) &= \exp \left\{ -\frac{i}{2} L_a \omega^a \right\} \psi(\mathbf{r}) \\ &=: U(\omega) \psi(\mathbf{r}).\end{aligned}\quad (15)$$

and leave (12) invariant. As will be discussed in more detail in section 3 the Gauß law is the infinitesimal generator of the residual gauge transformations (14), (14), (16) of homotopy class zero [36]. In order to further investigate its meaning we decompose the fields

into transversal and longitudinal parts

$$A_k^a = A_{t,k}^a + A_{\ell,k}^a; \quad E_k^a = E_{t,k}^a + E_{\ell,k}^a. \quad (16)$$

We observe that for the transversal fields the relations $\nabla \cdot \mathbf{E}_t^a = \nabla \cdot \mathbf{A}_t^a = 0$ hold, and from these relations it follows

$$\nabla \cdot \mathbf{E}_\ell^a + g f_{abc} \mathbf{A}^b \cdot \mathbf{E}^c = j_a^0. \quad (17)$$

The corresponding Hamilton operator can be written as

$$\begin{aligned}H &= \frac{1}{2} \int d^3r [E_{t,k}^a E_{t,k}^a + B_k^a B_k^a] \\ &\quad + \frac{1}{2} \int d^3r E_{\ell,k}^a E_{\ell,k}^a \\ &\quad + \int d^3r \psi^\dagger \left(\frac{1}{i} \vec{\alpha} \cdot \nabla + m\beta \right) \psi \\ &\quad + \frac{1}{2} \int d^3r g \psi^\dagger \vec{\alpha} \cdot L_a (\mathbf{A}_t^a + \mathbf{A}_\ell^a) \psi.\end{aligned}\quad (18)$$

It was demonstrated in I that by suitable gauge transformation for abelian theories, i.e. quantum electrodynamics, this Hamiltonian can be transformed into the Coulomb gauge Hamiltonian. In this way the Gauß constraint can be resolved on the classical level and the redundant variables can be eliminated. If one tries to extend this method to nonabelian theories one gets into difficulties. For brevity we give only the result of the corresponding nonabelian gauge transformations. By means of these transformations for $t = 0$ the Hamiltonian (18) goes over into

$$\begin{aligned}H &= \frac{1}{2} \int d^3r [\mathbf{E}_t^a \cdot \mathbf{E}_t^a + \mathbf{B}^a \cdot \mathbf{B}^a]_{/\mathbf{A}_\ell^a \equiv 0} \\ &\quad + \frac{1}{2} \int J_0^a(\mathbf{r}, 0) C(\mathbf{r} - \mathbf{r}') J_0^a(\mathbf{r}', 0) d^3r d^3r'_{/\mathbf{A}_\ell^a \equiv 0} \\ &\quad + \int d^3r \psi^\dagger \left(\frac{1}{i} \vec{\alpha} \cdot \nabla + m\beta + \frac{1}{2} g \vec{\alpha} L_a \cdot \mathbf{A}_t^a \right) \psi\end{aligned}\quad (19)$$

with $C(\mathbf{r}) := |\mathbf{r}|^{-1}$. Hence by the second term the Coulomb forces explicitly appear in the theory. It should be emphasized, however, that the appearance of Coulomb forces in the nonabelian theory only formally establishes an analogy to quantum electrodynamics. Namely in contrast to quantum electrodynamics the nonabelian charge densities $J_0^a(\mathbf{r}, 0)$ them-

selves contain the longitudinal fields, since in temporal gauge $J_0^a(\mathbf{r}, 0)$ is given by

$$\begin{aligned} J_0^a &= -gf_{abc}A^{bk}\partial_0 A_k^c + g\bar{\psi}L_a\gamma_0\psi \\ &= gf_{abc}A^{bk}E_k^c + g\bar{\psi}L_a\gamma_0\psi. \end{aligned} \quad (20)$$

Therefore, apart from the explicit manifestation of the Coulomb forces, for the explicit solution of the classical theory nothing is gained, i.e. it is not possible to remove the superfluous variables from the theory.

It would be a real progress if one were able to resolve the nonlinear Gauß law (17) with respect to the longitudinal electric field E_ℓ^a . But it is supposed that even if one succeeds to do so this would lead to chaos in the classical theory [37]. It is the aim of this paper to demonstrate that these difficulties can only be circumvented by an appropriate algebraic treatment of the corresponding quantized theory.

3. Nonabelian field quantization

In analogy to the abelian case we start with the temporal gauge $A_0^a \equiv 0$. The commutation relations of the nonabelian fields are then given by

$$\left[A_k^a(\mathbf{r}, t), E_{k'}^{a'}(\mathbf{r}', t) \right]_- = -i\delta_{aa'}\delta_{kk'}\delta(\mathbf{r} - \mathbf{r}') \quad (21)$$

while for the fermion fields we stipulate

$$\begin{aligned} &\left[\psi_{\alpha\Lambda}^\beta(\mathbf{r}, t), \psi_{\alpha'\Lambda'}^{\beta'}(\mathbf{r}', t) \right]_+ \\ &= (C\gamma^0)_{\alpha\alpha'}\sigma_{\Lambda\Lambda'}^1\delta(\mathbf{r} - \mathbf{r}')\delta_{\beta\beta'}, \end{aligned} \quad (22)$$

where the indices β and β' , respectively, refer to the fermion nonabelian representation. All other commutation or anticommutation relations vanish. Like in quantum electrodynamics it is our aim to incorporate the nonabelian Gauß law (6) into the dynamics. But as we shall see, it is not possible to achieve this incorporation by a gauge transformation. Nevertheless, the gauge transformations are a valuable tool for analyzing the nonabelian theory.

We consider residual gauge transformations of homotopy class zero of the nonabelian field in the linear state space of the theory to be given by $U = U_B U_F$ with

$$U_B := \exp \left\{ -i \int \omega_a(\mathbf{r}) \Gamma_a(\mathbf{A}, \mathbf{E}) d^3r \right\} \quad (23)$$

with

$$\Gamma_a(\mathbf{A}, \mathbf{E}) := (\nabla \delta_{ac} + gf_{abc}\mathbf{A}_b)\mathbf{E}_c \quad (24)$$

and

$$U_F := \exp \left\{ i \int \omega_a(\mathbf{r}) j_a^0(\mathbf{r}) d^3r \right\}, \quad (25)$$

where $j_a(\mathbf{r})$ is defined by (7). Homotopy classes unequal zero are omitted, as corresponding theta-angles unequal zero are not allowed to appear [38].

If gauge transformations are to be implemented in a linear space, they are to be subjected to the correspondence principle as any other transformation. That this principle is satisfied can be seen by direct calculation. One obtains with (23) - (25)

$$\begin{aligned} \frac{1}{2}L_a\mathbf{A}'^a &= \frac{1}{2}UL_a\mathbf{A}^aU^{-1} \\ &= \frac{1}{2}U(\omega)L_a\mathbf{A}^aU^{-1}(\omega) \\ &\quad - \frac{i}{g}[\nabla U(\omega)]U^{-1}(\omega), \end{aligned} \quad (26)$$

$$\begin{aligned} L_a\mathbf{E}'^a &= UL_a\mathbf{E}^aU^{-1} = U(\omega)L_a\mathbf{E}^aU^{-1}(\omega), \\ \psi'(\mathbf{r}) &= U\psi(\mathbf{r})U^{-1} = U(\omega)\psi(\mathbf{r}). \end{aligned} \quad (27) \quad (28)$$

Due to these formulas one can directly verify the gauge invariance of H against the residual gauge transformations $U = U_B U_F$, i.e. we have

$$H' = U H U^{-1} = H \quad (29)$$

or $[H, U]_- = 0$. Thus H and U must have common eigenstates.

However, as finite transformations U explicitly depend on the gauge functions $\omega_a(\mathbf{r})$, they are not suited for the definition and calculation of common eigenstates. This problem is solved by considering only infinitesimal transformations which can be characterized by infinitesimal generators. For $U = U_B U_F$ these generators are given by

$$G_a(\mathbf{r}, t) := \Gamma_a(\mathbf{A}, \mathbf{E}) - j_a^0(\mathbf{r}, t) \quad (30)$$

and it follows from (29) that also

$$[H, G_a(\mathbf{r}, t)]_- = 0, \quad a = 1 \dots 8 \quad (31)$$

must hold. Hence instead of U the set $\{G_a\}$, $a = 1, \dots, 8$ can be used to characterize the eigenstates of the theory.

These generators fulfill the commutation relations [36], [22]

$$[G_a(\mathbf{r}, t), G_b(\mathbf{r}', t)]_- = -if_{abc}G_c(\mathbf{r}, t)\delta(\mathbf{r} - \mathbf{r}'), \quad (32)$$

$$[G_a(\mathbf{r}, t), H]_- = 0. \quad (33)$$

This means that not all generators can simultaneously be diagonalized. Rather only a maximal set of commuting generators have simultaneous eigenstates and good quantum numbers, and these states and quantum numbers are compatible with the eigenstates of H , as all generators commute with H . But due to the invariance of H against all gauge transformations, all these transformations are compatible with the dynamics even if they do not possess common eigenstates. In addition, due to the gauge principle, for a subset of eigenstates one can indeed obtain common eigenstates for all generators, as can be concluded from the following argument.

According to the gauge principle all observable quantities have to be gauge invariant. If one extends this condition to the state space of a quantum theory, the states $|a\rangle$ must be invariant under gauge transformations, i.e., among all possible representations of the gauge group only singlet states are physically meaningful.

A singlet state is invariant under all group operations. This can be expressed by

$$G_a(\mathbf{r}, t)|a\rangle_{\text{singl.}} = 0, \quad a = 1, \dots, 8 \quad (34)$$

and means that in this special case the set of equations is compatible.

For the successful application of the algebraic formalism the gauge transformations $U = U_B U_F$ have to be further specified. In particular the gauge transformations have to be represented by unitary group operations [39]. The unitary implementation of U is guaranteed if the infinitesimal generators are hermitian operators. The latter property can be achieved by a suitable representation of the field operators which itself guarantees the hermiticity of these operators. This hermitian representation of field operators is well known from quantum electrodynamics and holds beyond perturbation theory. It can be transferred to non-abelian fields without essential modification. Hence for the following we assume the field operators in

$G_a(\mathbf{r}, t)$ to be hermitian operators with the consequence that the $G_a(\mathbf{r}, t)$ themselves become hermitian operators and U becomes unitarily implemented.

4. Nonabelian quantum field dynamics

For a more detailed discussion of the algebraic approach we refer to I and [40]. Here we repeat only some essentials.

The basis of the algebraic evaluation of quantum field dynamics is given by the Heisenberg relation for eigenstates $\langle 0|$ and $|a\rangle$ of H

$$\Delta E \langle 0|O|a\rangle = \langle 0|[O, H]_-|a\rangle, \quad (35)$$

where O is any element of the field operator algebra. This relation is only applicable to the genuine dynamical equations, but not to the constraints. In particular if we work in temporal gauge with the Gauß law as constraint, we have to show that nevertheless the Gauß law can be included into this formalism.

Following previous investigations the algebra elements O can be chosen as monomials of field operators in order to describe the dynamics of the quantized fields. In our case these monomials are given by

$$O := \mathcal{A}(\psi_{I_1} \dots \psi_{I_n})_t \mathcal{S}(B_{K_1} \dots B_{K_m})_t, \quad (36)$$

$$n, m = 0, 1, \dots, \infty,$$

where ψ_I are superspinors and B_K are superfields, $\psi := \{\psi, \psi^c\}$, $B := \{A^a, E^a\}$. The monomials (36) serve as algebraic basis elements for the construction of the GNS-basis states, and the projections of the basis states on a physical state vector $|a\rangle$ are integrated into a generating functional state. This functional state is defined by

$$|\mathcal{G}(j, b, a)\rangle := \sum_{n,m} \frac{i^n}{n!} \frac{i^m}{m!} \langle 0|\mathcal{A}(\psi_{I_1} \dots \psi_{I_n}) \cdot \mathcal{S}(B_{K_1} \dots B_{K_m})|a\rangle j_{I_1} \dots j_{I_n} b_{K_1} \dots b_{K_m} |0\rangle_F, \quad (37)$$

where j_I and b_K are the functional source operators for fermions or bosons, respectively.

In this formal description the field equations (9), (10) and (11) read

$$i\dot{\psi}_I = D_{II'}\psi_{I'} + W_{II'}^K B_K \psi_{I'} \quad (38)$$

$$i\dot{B}_K = L_{KK'} B_{K'} + M_K^{K_1 K_2} B_{K_1} B_{K_2} + V_K^{K_1 K_2 K_3} B_{K_1} B_{K_2} B_{K_3} + J_{II'}^K \psi_I \psi_{I'} \quad (39)$$

and the Hamiltonian (12) is given by

$$\begin{aligned}
 H = & \frac{1}{2} A_{I_1 I} D_{I I_2} \psi_{I_1} \psi_{I_2} + \frac{1}{2} A_{I_1 I} W_{I I_2}^K B_{K'} \psi_{I_1} \psi_{I_2} \\
 & + \frac{1}{2} C_{K_1 K} L_{K K_2} B_{K_1} B_{K_2} \\
 & + \frac{1}{3} C_{K_1 K} M_K^{K_2 K_3} B_{K_1} B_{K_2} B_{K_3} \\
 & + \frac{1}{4} C_{K_1 K} V_K^{K_2 K_3 K_4} B_{K_1} B_{K_2} B_{K_3} B_{K_4},
 \end{aligned} \quad (40)$$

where we applied the following definitions

$$D_{I I'} := -(i\alpha^k \partial_k - \beta m)_{\alpha\alpha'} \quad (41)$$

$$\begin{aligned}
 W_{I I'}^K &:= g\alpha_{\alpha\alpha'}^j \sigma_{\lambda\lambda'}^3 (\hat{L}_a)_{AA'}^{\lambda''\lambda'} \\
 &\cdot \delta_{1\eta} \delta(\mathbf{r} - \mathbf{r}') \delta(\mathbf{r} - \mathbf{z}),
 \end{aligned} \quad (42)$$

$$\begin{aligned}
 L_{K K'} &:= i\delta_{jj'} \delta_{aa'} \delta_{\eta 1} \delta_{2\eta'} \delta(\mathbf{z} - \mathbf{z}') \\
 &+ i\delta_{jj'} \delta_{aa'} \delta_{\eta 2} \delta_{1\eta'} \Delta_{\mathbf{z}} P_{\mathbf{z}}^{\mathbf{z}'} \delta(\mathbf{z} - \mathbf{z}'),
 \end{aligned} \quad (43)$$

$$\begin{aligned}
 M_K^{K_1 K_2} &:= -ig \left[\delta_{\eta 2} \delta_{1\eta_1} \delta_{1\eta_2} \right. \\
 &\cdot \left\{ f_{aa_1 a_2} \delta_{jj_2} \delta(\mathbf{z} - \mathbf{z}_2) \partial_{j_1}^{\mathbf{z}} \delta(\mathbf{z} - \mathbf{z}_1) \right. \\
 &+ 2f_{aa_1 a_2} \delta_{jj_2} \delta(\mathbf{z} - \mathbf{z}_1) \partial_{j_1}^{\mathbf{z}} \delta(\mathbf{z} - \mathbf{z}_2) \\
 &\left. \left. - f_{aa_1 a_2} \delta_{j_1 j_2} \delta(\mathbf{z} - \mathbf{z}_1) \partial_j^{\mathbf{z}} \delta(\mathbf{z} - \mathbf{z}_2) \right\} \right]_{\text{sym}(1,2)},
 \end{aligned} \quad (44)$$

$$\begin{aligned}
 V_K^{K_1 K_2 K_3} &:= \left[ig^2 \delta_{2\eta} \delta_{1\eta_1} \delta_{1\eta_2} \delta_{1\eta_3} f_{aa_1 b} f_{ba_2 a_3} \right. \\
 &\cdot \left. \delta_{j_1 j_2} \delta_{j_2 j_3} \delta(\mathbf{z} - \mathbf{z}_1) \delta(\mathbf{z} - \mathbf{z}_2) \delta(\mathbf{z} - \mathbf{z}_3) \right]_{\text{sym}(1,2,3)},
 \end{aligned} \quad (45)$$

$$\begin{aligned}
 J_{I I'}^K &:= -\frac{g}{2} (C\gamma^k)_{\alpha\alpha'} \sigma_{\lambda\lambda'}^2 (L_a)_{AA'}^{\lambda''\lambda'} \\
 &\cdot \delta(\mathbf{z} - \mathbf{r}) \delta(\mathbf{z} - \mathbf{r}') \delta_{2\eta}
 \end{aligned} \quad (46)$$

$$(\hat{L}_a)_{AA'}^{\lambda\lambda'} := \delta_{\lambda 1} (L_a)_{AA'} \delta_{1\lambda'} + \delta_{\lambda 2} (\hat{L}_a^t)_{AA'} \delta_{2\lambda'} \quad (47)$$

$$A_{I I'} := (C\gamma^0)_{\alpha\alpha'} \sigma_{\lambda\lambda'}^1 \delta(\mathbf{r} - \mathbf{r}') \delta_{AA'}, \quad (48)$$

$$C_{K K'} := -\delta_{jj'} \sigma_{\eta\eta'}^2 \delta_{aa'} \delta(\mathbf{z} - \mathbf{z}') \quad (49)$$

and where the indices I and K mean

$$I := \mathbf{r}, \alpha, \lambda, A; \quad K := \mathbf{z}, j, a, \eta \quad (50)$$

with \mathbf{r} = position, α = spinor index, λ = superspinor index, $\lambda = 1 \equiv$ spinor, $\lambda = 2 \equiv$ charge conjugated spinor, and \mathbf{z} = position, j = vector index, $A, a =$

nonabelian group index, $\eta = 1 =$ vector potential, $\eta = 2 =$ color electric field.

In this form, by means of (35), (36), (37) the field equations (38), (39) can be mapped into functional space. We employ two theorems describing the rearrangement of unsymmetrical monomials of field operators into series of symmetrical monomials for boson operators and antisymmetrical ones for fermion operators and which are expressed in terms of generating functional states. These theorems are formulated in I and we do not explicitly repeat them, rather we refer to I and the proofs given in the papers cited there.

These theorems directly allow the mapping of (35) for all monomials O into functionals space. In so doing we obtain for any $\langle 0|(OH)|a\rangle$ and $\langle 0|(HO)|a\rangle$ a projection and if the set of O is complete this is equivalent to the functional equation

$$\begin{aligned}
 &\left\{ H \left(\frac{1}{i} \partial^b - \frac{i}{2} Cb, \frac{1}{i} \partial^f - \frac{i}{2} Aj \right) \right. \\
 &\quad \left. - H \left(\frac{1}{i} \partial^b + \frac{i}{2} Cb, \frac{1}{i} \partial^f + \frac{i}{2} Aj \right) \right\} |\mathcal{G}\rangle = \Delta E |\mathcal{G}\rangle,
 \end{aligned} \quad (51)$$

where $H \equiv H(B, \psi)$ in accordance with (40).

This kind of transformation of Hilbert space equations into functional equations can be equally well applied to the Gauß constraint (6).

In our formal description the classical nonabelian Gauß law reads

$$\begin{aligned}
 G^L(B, \psi) &:= N_K^L B_K + S_{K_1 K_2}^L B_{K_1} B_{K_2} \\
 &+ R_{I_1 I_2}^L \psi_{I_1} \psi_{I_2} = 0
 \end{aligned} \quad (52)$$

with

$$N_{K_1}^L := \delta_{aa_1} \delta_{\eta 1} \partial_{j_1}^{\mathbf{r}} \delta(\mathbf{r} - \mathbf{r}_1) \quad (53)$$

$$\begin{aligned}
 S_{K_1 K_2}^L &:= -g \left[f_{aa_1 a_2} \delta_{j_1 j_2} \delta(\mathbf{r} - \mathbf{r}_1) \right. \\
 &\cdot \left. \delta(\mathbf{r} - \mathbf{r}_2) \delta_{\eta 1} \delta_{\eta 2} \right]_{\text{sym}(K_1 K_2)},
 \end{aligned}$$

$$\begin{aligned}
 R_{I_1 I_2}^L &:= i \frac{g}{2} (C\gamma^0)_{\alpha_1 \alpha_2} \sigma_{\lambda_1 \lambda_2}^2 (\hat{L}_a)_{A_1 A_2}^{\lambda_3 \lambda_2} \\
 &\cdot \delta(\mathbf{r} - \mathbf{r}_1) \delta(\mathbf{r} - \mathbf{r}_2).
 \end{aligned}$$

In quantum chromodynamics the Gauß law is weakly implemented by $G^L(B, \psi)|a\rangle = 0$ for all physical states $|a\rangle$ in accordance with Equations (34).

In the algebraic version this weak condition is projected on the GNS-basis states, which leads to the set of equations

$$\langle 0 | \mathcal{A}(\psi_{I_1} \dots \psi_{I_n}) \mathcal{S}(B_{K_1} \dots B_{K_m}) G^L(B, \psi) | a \rangle = 0 \quad (54)$$

for $n, m = 0, 1 \dots \infty$.

For these projections the following theorem holds:

Proposition 1: For unitary gauge transformations U and gauge invariant states $|a\rangle$ the images of the Gauß constraints (34) in the functional space are given by the equations

$$G^L \left(\frac{1}{i} \partial^b - \frac{i}{2} Cb, \frac{1}{i} \partial^f - \frac{i}{2} Aj \right) | \mathcal{G} \rangle = 0, \quad \forall L, \quad (55)$$

and

$$\langle \mathcal{S} | G^L \left(\frac{1}{i} \partial^b - \frac{i}{2} Cb, \frac{1}{i} \partial^f - \frac{i}{2} Aj \right) = 0, \quad (56)$$

where $\langle \mathcal{S} |$ is the functional left solution

$$\langle \mathcal{S}(j, b, a) | := \sum_{n,m} \frac{i^n i^m}{n! m!} \sigma_{nm}^* (I_1 \dots I_n, K_1 \dots K_m) \cdot {}_F \langle 0 | \partial_{K_1}^b \dots \partial_{K_m}^b \partial_{I_1}^f \dots \partial_{I_n}^f \quad (57)$$

and $| \mathcal{G} \rangle$ is defined by (37).

Proof: With definition (36) we use the symbolic notation $|e_N\rangle := O_{nm}^+ |0\rangle$ for the GNS-states in Hilbert space. Then the states $|a\rangle$ can be represented by

$$|a\rangle = \sum_N \sigma^N |e_N\rangle, = \sum_N \tau_N |e^N\rangle \quad (58)$$

where $|e^N\rangle$ is the dual of $|e_N\rangle$, i.e., $\langle e^N | e_M \rangle = \delta_M^N$. Substitution of the dual representation of $|a\rangle$ into equations (34) and projection with $\langle e_M |$ yields

$$\sum_N \langle e_M | G^L | e^N \rangle \tau_N = 0. \quad (59)$$

Due to Proposition 2 of I this is equivalently expressed in functional space by equations (55).

For the derivation of (56) we consider the matrix elements $\langle a | G^L | e^N \rangle$. For unitary U the generators G^L must be hermitian. Hence it is $\langle a | G^L | e^N \rangle =$

$\langle G^L a | e^N \rangle = 0$ for singlett states. With (58) this can be expressed by

$$\sum_M \langle e_M | G^L | e^N \rangle (\sigma^M)^* = 0 \quad (60)$$

and if mapped in functional space one obtains (56). \diamond

Proposition 2: For unitary gauge transformations U and gauge invariant states $|a\rangle$, in addition to Eqs. (55), (56) also equations

$$G^L \left(\frac{1}{i} \partial^b + \frac{i}{2} Cb, \frac{1}{i} \partial^f + \frac{i}{2} Aj \right) | \mathcal{G} \rangle = 0, \quad \forall L \quad (61)$$

have to be simultaneously fulfilled.

Proof: Due to the hermiticity of G^L we have

$$\langle 0 | G^L O_{nm} | a \rangle = \langle G^L 0 | O_{nm} | a \rangle = 0 \quad (62)$$

as $\langle 0 |$ is gauge invariant. Application of proposition 1 of I to $\langle 0 | G^L O_{nm} | a \rangle$ yields equation (61) in functional space. \diamond

With respect to the left solutions proposition 2 is incomplete. We postpone the proof that also for (61) left solutions must exist until we explicitly evaluate equations (55), (56) and (61).

The basic variables in functional space are the fermionic sources j_I and the bosonic sources b_K . Together with their duals ∂_I^f and ∂_K^b the following nontrivial relations hold,

$$\begin{aligned} [j_I, \partial_{I'}^f]_+ &= \delta_{II'}, \\ [\partial_{K'}^b, b_K]_- &= \delta_{KK'} \end{aligned} \quad (63)$$

while all other commutators or anticommutators, respectively, vanish. The generating functionals (37) are embedded in a functional Fock space, i.e. $\partial_I^f |0\rangle_F = \partial_K^b |0\rangle_F = 0, \quad \forall I, K$. With respect to the following discussion it should be emphasized that this functional Fock space is only a book-keeping device and lacks any physical meaning. In particular this functional Fock space does not fix the field representations in various Hilbert spaces from the outset, because these representations are expressed by the matrix elements $\langle 0 | O | a \rangle$ or $\langle 0 | O | 0 \rangle$ themselves which are in no way influenced by this auxiliary Fock space. The only task

of this Fock space is to allow a compact formulation of the field dynamics governed by the field algebra.

From equations (51) and (55) it is obvious that not the sources j and b and their duals themselves are the variables of the functional Hamilton and Gauß operators. Rather these operators exclusively depend on “Bogoljubov” transformed functional variables, which of course are also only book-keeping devices! For these new variables the following nontrivial relations hold

$$\begin{aligned} & \left[\left(\frac{1}{i} \partial_K^b - \frac{i}{2} C_{KL} b_L \right), \left(\frac{1}{i} \partial_{K'}^b - \frac{i}{2} C_{K'L'} b_{L'} \right) \right]_- \quad (64) \\ & = C_{KK'} \mathbb{I}, \\ & \left[\left(\frac{1}{i} \partial_K^b + \frac{i}{2} C_{KL} b_L \right), \left(\frac{1}{i} \partial_{K'}^b + \frac{i}{2} C_{K'L'} b_{L'} \right) \right]_- \\ & = -C_{KK'} \mathbb{I}, \\ & \left[\left(\frac{1}{i} \partial_I^f - \frac{i}{2} A_{IJ} j_J \right), \left(\frac{1}{i} \partial_{I'}^f - \frac{i}{2} A_{I'J'} j_{J'} \right) \right]_+ \quad (65) \\ & = -A_{II'} \mathbb{I}, \\ & \left[\left(\frac{1}{i} \partial_I^f + \frac{i}{2} A_{IJ} j_J \right), \left(\frac{1}{i} \partial_{I'}^f + \frac{i}{2} A_{I'J'} j_{J'} \right) \right]_+ \\ & = A_{II'} \mathbb{I} \end{aligned}$$

while *all* other commutators or anticommutators vanish.

We notice that in Hilbert space the quantization conditions (21), (22) can be rewritten in terms of the formal fields B_K and ψ_I which leads to

$$\begin{aligned} [\psi_I, \psi_{I'}]_+ &= A_{II'} \mathbb{I}, \\ [B_K, B_{K'}]_- &= C_{KK'} \mathbb{I} \end{aligned} \quad (66)$$

A comparison with (65), (66) shows that the “Bogoljubov” transforms of j_I and b_K obey isomorphic commutation or anticommutation relations to their corresponding original fields, i.e., their inverse images in Hilbert space. From this it follows immediately that the images of H and G in functional space obey the same commutation rules as their inverse images in Hilbert space, see Proposition 3 in I.

5. Isomorphism of fields and sources

In this section we study the consequences of the new functional variables introduced in the preceding section. Although these combinations follow directly from the rearrangement theorems, it is convenient to

introduce a slightly modified version. This was discussed in detail in I and we do not repeat it here, but refer to I. We define the new variables

$$X_{j,a}^1(\mathbf{r}) := \frac{1}{i} \partial_{j,a}^A(\mathbf{r}) + \frac{1}{2} b_{j,a}^E(\mathbf{r}) =: X_K^1, \quad (67)$$

$$Y_{j,a}^1(\mathbf{r}) := \partial_{j,a}^E(\mathbf{r}) - \frac{i}{2} b_{j,a}^A(\mathbf{r}) =: Y_K^1,$$

$$\partial_{\alpha,A}^1(\mathbf{r}) := \frac{1}{i} \partial_{\alpha,A}(\mathbf{r}) - \frac{i}{2} (C\gamma^0)_{\alpha\beta} j_{\beta,A}^c(\mathbf{r}) =: \partial_I^1,$$

$$Z_{\alpha,A}^1(\mathbf{r}) := -\frac{1}{i} (C\gamma^0)_{\alpha\beta} \partial_{\beta,A}^c(\mathbf{r}) + \frac{i}{2} j_{\alpha,A}(\mathbf{r}) =: Z_I^1,$$

and

$$X_{j,a}^2(\mathbf{r}) := \frac{1}{i} \partial_{j,a}^A(\mathbf{r}) - \frac{1}{2} b_{j,a}^E(\mathbf{r}) =: X_K^2, \quad (68)$$

$$Y_{j,a}^2(\mathbf{r}) := \partial_{j,a}^E(\mathbf{r}) + \frac{i}{2} b_{j,a}^A(\mathbf{r}) =: Y_K^2,$$

$$\partial_{\alpha,A}^2(\mathbf{r}) := \frac{1}{i} \partial_{\alpha,A}(\mathbf{r}) + \frac{i}{2} (C\gamma^0)_{\alpha\beta} j_{\beta,A}^c(\mathbf{r}) =: \partial_I^2,$$

$$Z_{\alpha,A}^2(\mathbf{r}) := \frac{1}{i} (C\gamma^0)_{\alpha\beta} \partial_{\beta,A}^c(\mathbf{r}) + \frac{i}{2} j_{\alpha,A}(\mathbf{r}) =: Z_I^2.$$

They obey the commutation or anticommutation relations

$$[Y_K^1, X_{K'}^1]_- = \delta_{KK'}, \quad (69)$$

$$[Z_I^1, \partial_{I'}^1]_+ = \delta_{II'},$$

$$[X_K^2, Y_{K'}^2]_- = \delta_{KK'},$$

$$[Z_I^2, \partial_{I'}^2]_+ = \delta_{II'}$$

while all other commutators or anticommutator, respectively, vanish. The renormalized energy eigenvalue equation (51) in functional space then reads explicitly in these new variables

$$\begin{aligned} & \sum_{i=1}^2 \int \left\{ \frac{(-1)^i}{2} Y_{j,a}^i(\mathbf{r}) Y_{j,a}^i(\mathbf{r}) \right. \quad (70) \\ & - \frac{(-1)^i}{2} \left[\epsilon_{jhl} \partial_h X_{\ell a}^i(\mathbf{r}) + \frac{1}{2} \epsilon_{jhl} g f_{abc} X_{hb}^i(\mathbf{r}) X_{\ell c}^i(\mathbf{r}) \right] \\ & \cdot \left[\epsilon_{jh'l'} \partial_{h'} X_{\ell'a}^i(\mathbf{r}) + \frac{1}{2} \epsilon_{jh'l'} g f_{ab'c'} X_{h'b'}^i(\mathbf{r}) X_{\ell'c'}^i(\mathbf{r}) \right] \\ & + g X_{ja}^i(\mathbf{r}) Z_{\alpha A}^i(\mathbf{r}) \alpha_{\alpha\beta}^j L_{AB}^a \partial_{\beta B}^i(\mathbf{r}) \\ & \left. - Z_{\alpha A}^i(\mathbf{r}) (i\alpha^k \partial_k - \beta m)_{\alpha\beta} \partial_{\beta A}^i(\mathbf{r}) \right\} d^3r |\mathcal{G}\rangle = \Delta E |\mathcal{G}\rangle \end{aligned}$$

and the Gauß constraints (55) and (61) are given by

$$\left\{ \frac{1}{i} \partial_k Y_{ka}^i(\mathbf{r}) + i g f_{abc} X_{jb}^i(\mathbf{r}) Y_{jc}^i(\mathbf{r}) + (-1)^i g \left[Z_{\alpha A}^i(\mathbf{r}) L_{AB}^a \partial_{\alpha B}^i(\mathbf{r}) \right] \right\} |\mathcal{G}\rangle = 0. \quad (71)$$

A comparison with the Hamiltonian (40) in Hilbert space shows that the Hilbert space variables A are replaced by X_K^1, X_K^2 , the E -fields are replaced by Y_K^1, Y_K^2 , while Z^i, ∂^i correspond to ψ^c, ψ . The doubling of the degrees of freedom is no failure of the theory, because X_K^1, X_K^2 and so forth result by a canonical transformation from the original source operators j^A, ∂^A and so forth, and this canonical transformation is biunique. In addition, the fact that two Hamiltonians occur in equation (70) indicates that the Heisenberg equation (35) is expressed in functional space and not the eigenvalue equation for H itself. Nevertheless, in the following we will show that this similarity of the functional equation to Hamiltonian eigenvalue equations allows a derivation of Coulomb forces in temporal gauge.

We observe that in the commutation or anticommutation relations (69) the sequence of the $i = 2$ variables is reversed in comparison with the $i = 1$ variables. In comparison with the conventional commutation relations in Fock space this difference enforces a different interpretation of the $i = 2$ variables compared to the $i = 1$ variables. In order to clearly identify the meaning of the various variables in the following we use the notation

$$\begin{aligned} \hat{\partial}_K^1 &:= Y_K^1; \quad \hat{X}_K^1 := X_K^1, \\ \hat{\partial}_K^2 &:= X_K^2; \quad \hat{X}_K^2 := Y_K^2 \end{aligned} \quad (72)$$

which expresses the difference between these pairs of variables. However, the change of notation is not sufficient for a proper treatment of our transformation problem. Rather, together with this introduction of a new notation it is necessary to change the coordinate system in functional space in order to work successfully with the new variables.

It was shown in I that the transformations (67), (68) have to be supplied by the introduction of a new functional vacuum. If we denote the transformed functional vacuum by $|0\rangle'_F$ the operators $\hat{\partial}_K^1 := Y_K^1$ and $\hat{\partial}_K^2 := X_K^2$ should act as annihilation operators. This

means

$$\hat{\partial}_K^1 |0\rangle'_F = \left(\partial_K^E - \frac{i}{2} b_K^A \right) |0\rangle'_F \stackrel{!}{=} 0, \quad (73)$$

$$\hat{\partial}_K^2 |0\rangle'_F = \frac{1}{i} \left(\partial_K^A - \frac{i}{2} b_K^E \right) |0\rangle'_F \stackrel{!}{=} 0 \quad (74)$$

and analogous conditions for the fermion operators. In the case of nonabelian fields the new vacuum $|0\rangle'_F$ is related to the original functional vacuum state $|0\rangle_F$ by the transformation

$$|0\rangle'_F = \exp \left\{ \frac{i}{2} \int b_{ja}^A(\mathbf{r}) b_{ja}^E(\mathbf{r}) d^3r + \frac{1}{2} \int j_{\alpha A}^c(\mathbf{r}) (C\gamma^0)_{\alpha\beta} j_{\beta A}(\mathbf{r}) d^3r \right\} |0\rangle_F. \quad (75)$$

In the following we assume $|\mathcal{G}\rangle$ to be expressed in this transformed Fock basis and solve equations (70), (71) under this condition. In a first step we rewrite the eigenvalue equation (70) and the constraints in the new variables. We define

$$\begin{aligned} \mathcal{H}_1 &:= \int \left\{ -\frac{1}{2} \hat{\partial}_{ja}^1(\mathbf{r}) \hat{\partial}_{ja}^1(\mathbf{r}) + \frac{1}{2} \left[\epsilon_{jhl} \partial_h \hat{X}_{\ell a}^1(\mathbf{r}) + \frac{1}{2} \epsilon_{jhl} g f_{abc} \hat{X}_{hb}^1(\mathbf{r}) \hat{X}_{\ell c}^1(\mathbf{r}) \right] \right. \\ &\quad \cdot \left[\epsilon_{jh'\ell'} \partial_{h'} \hat{X}_{\ell'a}^1(\mathbf{r}) + \frac{1}{2} \epsilon_{jh'\ell'} g f_{ab'c'} \hat{X}_{h'b'}^1(\mathbf{r}) \hat{X}_{\ell'c'}^1(\mathbf{r}) \right] \\ &\quad + g \hat{X}_{ja}^1(\mathbf{r}) Z_{\alpha A}^1(\mathbf{r}) \alpha_{\alpha\beta}^j L_{AB}^a \partial_{\beta B}^1(\mathbf{r}) \\ &\quad \left. - Z_{\alpha A}^1(\mathbf{r}) (i\alpha^k \partial_k - \beta m)_{\alpha\beta} \partial_{\beta A}^1(\mathbf{r}) \right\} d^3r \end{aligned} \quad (76)$$

and

$$\begin{aligned} \mathcal{H}_2 &:= \int \left\{ -\frac{1}{2} \hat{X}_{ja}^2(\mathbf{r}) \hat{X}_{ja}^2(\mathbf{r}) + \frac{1}{2} \left[\epsilon_{jhl} \partial_h \hat{\partial}_{\ell a}^2(\mathbf{r}) + \frac{1}{2} \epsilon_{jhl} g f_{abc} \hat{\partial}_{hb}^2(\mathbf{r}) \hat{\partial}_{\ell c}^2(\mathbf{r}) \right] \right. \\ &\quad \cdot \left[\epsilon_{jh'\ell'} \partial_{h'} \hat{\partial}_{\ell'a}^2(\mathbf{r}) + \frac{1}{2} \epsilon_{jh'\ell'} g f_{ab'c'} \hat{\partial}_{h'b'}^2(\mathbf{r}) \hat{\partial}_{\ell'c'}^2(\mathbf{r}) \right] \\ &\quad - g \hat{\partial}_{ja}^2(\mathbf{r}) Z_{\alpha A}^2(\mathbf{r}) \alpha_{\alpha\beta}^j L_{AB}^a \partial_{\beta B}^2(\mathbf{r}) \\ &\quad \left. + Z_{\alpha A}^2(\mathbf{r}) (i\alpha^k \partial_k - \beta m)_{\alpha\beta} \partial_{\beta A}^2(\mathbf{r}) \right\} d^3r \end{aligned} \quad (77)$$

and thus from (70) we obtain the eigenvalue equation

$$(\mathcal{H}_1 - \mathcal{H}_2)|\mathcal{G}\rangle = \Delta E|\mathcal{G}\rangle \quad (78)$$

while the Gauß constraints (71) go over into

$$\left[i\partial_k \hat{\partial}_{ka}^1(\mathbf{r}) - igf_{abc} \hat{X}_{jb}^1(\mathbf{r}) \hat{\partial}_{jc}^1(\mathbf{r}) + \rho_a^1(\mathbf{r}) \right] |\mathcal{G}\rangle = 0, \quad (79)$$

$$\left[i\partial_k \hat{X}_{ka}^2(\mathbf{r}) - igf_{abc} \hat{\partial}_{jb}^2(\mathbf{r}) \hat{X}_{jc}^2(\mathbf{r}) - \rho_a^2(\mathbf{r}) \right] |\mathcal{G}\rangle = 0, \quad (80)$$

$$\rho_a^i(\mathbf{r}) := gZ_{\alpha A}^i(\mathbf{r}) L_{AB}^a \partial_{\alpha B}^i(\mathbf{r}). \quad (81)$$

We observe that due to the transformation (75) and the corresponding commutation relations for the new variables, the operators \hat{X}_K^i, Z_I^i are creation operators, while $\hat{\partial}_K^i, \partial_I^i$ are annihilation operators. Hence we have the relations

$$\begin{aligned} \hat{\partial}_K^i &\equiv \left(\hat{X}_K^i \right)^+ \\ \partial_I^i &\equiv \left(Z_I^i \right)^+ \end{aligned}, \quad i = 1, 2 \quad (82)$$

and (78), (79), (80) can be written in the following form

$$\left[\mathcal{H}_1 \left((\hat{X}^1)^+, \hat{X}^1, (Z^1)^+, Z^1 \right) - \mathcal{H}_2 \left((\hat{X}^2)^+, \hat{X}^2, (Z^2)^+, Z^2 \right) \right] |\mathcal{G}\rangle = \Delta E|\mathcal{G}\rangle, \quad (83)$$

$$\left[i\partial_k \hat{X}_{ka}^1(\mathbf{r})^+ - igf_{abc} \hat{X}_{jb}^1(\mathbf{r}) \hat{X}_{jc}^1(\mathbf{r})^+ + \rho_a^1(\mathbf{r}) \right] |\mathcal{G}\rangle = 0, \quad (84)$$

$$\left[i\partial_k \hat{X}_{ka}^2(\mathbf{r}) - igf_{abc} \hat{X}_{jb}^2(\mathbf{r})^+ \hat{X}_{jc}^2(\mathbf{r}) - \rho_a^2(\mathbf{r}) \right] |\mathcal{G}\rangle = 0. \quad (85)$$

From these equations it follows that in (83) - (85) the two sets of variables $i = 1, 2$ are strictly separated and that in addition the Fock vacuum $|0\rangle_F$ can be represented by $|0\rangle_F = |\hat{0}\rangle_F^1 \otimes |\hat{0}\rangle_F^2$, i.e. the direct product of the vacua for the two sets of variables. In accordance with these properties we can introduce the ansatz

$$|\mathcal{G}\rangle = |\mathcal{G}\rangle^1 \otimes |\mathcal{G}\rangle^2, \quad (86)$$

where the states $|\mathcal{G}\rangle^i$, $i = 1, 2$ are referred to the corresponding subspaces of the whole Fock space. These states then have to satisfy the equations

$$\mathcal{H}_1|\mathcal{G}\rangle^1 = E_1|\mathcal{G}\rangle^1, \quad \mathcal{H}_2|\mathcal{G}\rangle^2 = E_2|\mathcal{G}\rangle^2, \quad (87)$$

$$\left[i\partial_k \hat{X}_{ka}^1(\mathbf{r})^+ - igf_{abc} \hat{X}_{jb}^1(\mathbf{r}) \hat{X}_{jc}^1(\mathbf{r})^+ + \rho_a^1(\mathbf{r}) \right] |\mathcal{G}\rangle^1 = 0, \quad (88)$$

$$\left[i\partial_k \hat{X}_{ka}^2(\mathbf{r}) - igf_{abc} \hat{X}_{jb}^2(\mathbf{r})^+ \hat{X}_{jc}^2(\mathbf{r}) - \rho_a^2(\mathbf{r}) \right] |\mathcal{G}\rangle^2 = 0. \quad (89)$$

Hence in the following we have to evaluate Eqs. (87) - (90). As in these equations absolute energy values appear we consider the whole system enclosed in a finite volume V . Afterwards we remove this restriction and perform the limit $V \rightarrow \infty$. We consider the cases $i = 1, 2$ separately and later on combine the results.

We first show that it is not necessary to treat the case $i = 2$ explicitly. Once the $i = 1$ case is solved the results can immediately be taken over to the $i = 2$ case. For this case the corresponding equations are

$$\mathcal{H}_2|\mathcal{G}\rangle^2 = E_2|\mathcal{G}\rangle^2, \quad (90)$$

$$\left[i\partial_k \hat{X}_{ka}^2(\mathbf{r}) - igf_{abc} \hat{X}_{jb}^2(\mathbf{r})^+ \hat{X}_{jc}^2(\mathbf{r}) - \rho_a^2(\mathbf{r}) \right] |\mathcal{G}\rangle^2 = 0.$$

By proposition 2 the existence of (90) and their right solutions is secured, provided the corresponding states $|a\rangle$ in Hilbert space exist. But in contrast to proposition 1 no statement about the left solutions was made. The explicit form of the equations given above allows to solve this problem.

Proposition 3: Equations (90) possess left solutions.

Proof: If left solutions ${}^2\langle S|$ of equations (90) exist, they have to fulfill the equations

$${}^2\langle S|\mathcal{H}_2 = {}^2\langle S|E_2, \quad (91)$$

$${}^2\langle S|\left[i\partial_k \hat{X}_{ka}^2(\mathbf{r}) - igf_{abc} \hat{X}_{jb}^2(\mathbf{r})^+ \hat{X}_{jc}^2(\mathbf{r}) - \rho_a^2(\mathbf{r}) \right] = 0.$$

By hermitian conjugation in functional space we obtain from (91)

$$\mathcal{H}_2^\dagger|S\rangle^2 = E_2|S\rangle^2, \quad (92)$$

$$\left[i\partial_k \hat{X}_{ka}^2(\mathbf{r})^+ - igf_{abc} \hat{X}_{jb}^2(\mathbf{r}) \hat{X}_{jc}^2(\mathbf{r})^+ + \rho_a^2(\mathbf{r})^+ \right] |S\rangle^2 = 0.$$

From (77), (82) and (89) it follows that (92) are analogous to the corresponding equations for the $i = 1$ case if one replaces $i = 1$ by $i = 2$. This means that (92) are isomorphic to the corresponding equations for $|\mathcal{G}\rangle^1$ and that by substitution $\hat{X}^1 \rightarrow \hat{X}^2$ the state $|\mathcal{G}\rangle^1$ can be mapped upon $|S\rangle^2$. Hence $|S\rangle^2$ exists in accordance with proposition 1. As in functional space hermitian conjugation is well defined, it follows from the existence of (92) that also (91) exists. \diamond

Thus in the following we concentrate on the treatment of the $i = 1$ case. In particular the Gauß constraints (89) read

$$\left\{ i\partial_k \hat{\partial}_{a,k}^1(\mathbf{r}) - igf_{aa_1a_2} \hat{X}_{a_1j}^1(\mathbf{r}) \hat{\partial}_{a_2j}^1(\mathbf{r}) + \rho_a^1(\mathbf{r}) \right\} |\mathcal{G}\rangle^1 = 0.$$

$$(93)$$

The divergence operation in (93) acts on the functional variables and simultaneously on fields in Hilbert space as a projection operator which annihilates the transversal parts of $\hat{\partial}_{a,k}^1(\mathbf{r})$. For a proper treatment of (93) we thus have to decompose all vector variables in their longitudinal and transversal branches. This decomposition for functional variables can be performed in the same way as in function space and we obtain

$$\begin{aligned}\hat{\partial}_K^1 &= (P_\ell + P_t) \hat{\partial}_K^1 := \hat{\partial}_K^{1,\ell} + \hat{\partial}_K^{1,t}, \\ \hat{X}_K^1 &= (P_\ell + P_t) \hat{X}_K^1 := \hat{X}_K^{1,\ell} + \hat{X}_K^{1,t}\end{aligned}\quad (94)$$

with P_ℓ and P_t projectors in function space. With this decomposition the commutation relations (69) are transformed into

$$\left[\hat{\partial}_K^{1,\ell}, \hat{X}_{K'}^{1,\ell} \right]_- = iP_t \delta_{KK'}, \quad (95)$$

$$\left[\hat{\partial}_K^{1,t}, \hat{X}_{K'}^{1,t} \right]_- = iP_t \delta_{KK'} \quad (96)$$

and (93) goes over into

$$\begin{aligned}\left\{ i\partial_k \hat{\partial}_{a,k}^{1,\ell}(\mathbf{r}) - igf_{abc} \hat{X}_{b,k}^1(\mathbf{r}) \hat{\partial}_{c,k}^{1,\ell}(\mathbf{r}) \right. \\ \left. - igf_{abc} \hat{X}_{b,k}^1(\mathbf{r}) \hat{\partial}_{c,k}^{1,t}(\mathbf{r}) + \rho_a^1(\mathbf{r}) \right\} |\mathcal{G}\rangle^1 = 0.\end{aligned}\quad (97)$$

Simultaneously with this operator decomposition the functional Fock space representation of $|\mathcal{G}\rangle$ in the new variables \hat{X}_K^1 , \hat{X}_K^2 , etc., can be rewritten in terms of longitudinal and transversal operators.

6. Resolution of the Gauß constraint

In the preceding section we demonstrated that in the new variables (67), (68) the functional equation (51) for the renormalized energy can be decomposed into two independent energy equations (87) and corresponding Gauß constraints (89), (90). Furthermore, it was shown that it suffices to treat the $i = 1$ case explicitly, while the $i = 2$ case can be treated along the same lines, i.e., the results of the $i = 1$ case can be taken over for the $i = 2$ case. Hence in the following we concentrate on the $i = 1$ case.

The variables of the $i = 1$ case are given by (67). With respect to these variables we have the following

correspondence: The classical \mathbf{A} -fields correspond to X_K^1 variables, while classical \mathbf{E} -fields correspond to Y_K^1 variables. This correspondence leads to the functional Hamiltonian \mathcal{H}_1 given in (76). With the decomposition (94) we obtain

$$\begin{aligned}\int \hat{\partial}_{ja}^1(\mathbf{r}) \hat{\partial}_{ja}^1(\mathbf{r}) d^3r &= \int \hat{\partial}_{ja}^{1,\ell}(\mathbf{r}) \hat{\partial}_{ja}^{1,\ell}(\mathbf{r}) d^3r \\ &+ \int \hat{\partial}_{ja}^{1,t}(\mathbf{r}) \hat{\partial}_{ja}^{1,t}(\mathbf{r}) d^3r\end{aligned}\quad (98)$$

and thus \mathcal{H}_1 admits the representation

$$\begin{aligned}\mathcal{H}_1 &= -\frac{1}{2} \int \hat{\partial}_{a,k}^{1,\ell}(\mathbf{r}) \hat{\partial}_{a,k}^{1,\ell}(\mathbf{r}) d^3r \\ &+ h_1 \left(\hat{\partial}^{1,t}, \hat{X}^{1,\ell}, \hat{X}^{1,t}, Z^1, \partial^1 \right),\end{aligned}\quad (99)$$

where the first term represents the longitudinal part of the color electric energy, while the second term does not contain $\hat{\partial}^{1,\ell}$ anymore, i.e., is independent of the color electric field. Thus we can proceed in analogy to quantum electrodynamics; i.e., the first term of (99) has to be transformed into the color electric nonabelian ‘‘Coulomb’’ law, where of course we expect drastic modifications compared to quantum electrodynamics.

As the first step we evaluate the nonabelian Gauß constraint (93). We substitute $\hat{\partial}_{a,k}^{1,\ell}(\mathbf{r}) = \partial_k^r \partial_a^1(\mathbf{r})$ into (97) and after multiplication with Δ^{-1} this yields

$$\begin{aligned}\left\{ \partial_a^1(\mathbf{r}) + gf_{abc} \frac{1}{4\pi} \int C(\mathbf{r}, \mathbf{r}') \hat{X}_{b,j}^1(\mathbf{r}') \partial_j^{r'} \partial_c^1(\mathbf{r}') d^3r' \right. \\ \left. + \frac{1}{4\pi} \int C(\mathbf{r}, \mathbf{r}') \hat{\rho}_a^1(\mathbf{r}') d^3r' \right\} |\mathcal{G}\rangle^1 = 0\end{aligned}\quad (100)$$

with

$$\hat{\rho}_a^1(\mathbf{r}) := gf_{abc} \hat{X}_{b,j}^1(\mathbf{r}) \hat{\partial}_{c,j}^{1,t}(\mathbf{r}) + i\rho_a^1(\mathbf{r}). \quad (101)$$

We rewrite equation (100) in the form

$$\begin{aligned}\int \left[\delta_{ac} \delta(\mathbf{r} - \mathbf{r}') + g \frac{1}{4\pi} f_{abc} C(\mathbf{r}, \mathbf{r}') \hat{X}_{b,j}^1(\mathbf{r}') \partial_j^{r'} \right] \\ \cdot \partial_c^1(\mathbf{r}') d^3r' |\mathcal{G}\rangle^1 = -\frac{1}{4\pi} \int C(\mathbf{r}, \mathbf{r}') \hat{\rho}_a^1(\mathbf{r}') d^3r' |\mathcal{G}\rangle^1\end{aligned}\quad (102)$$

and in order to resolve this local Gauß law we have

to construct the resolvent of the operator

$$K_{ac}(\mathbf{r}, \mathbf{r}') := \left[\delta_{ac} \delta(\mathbf{r} - \mathbf{r}') + g \frac{1}{4\pi} f_{abc} C(\mathbf{r}, \mathbf{r}') \hat{X}_{b,j}^1(\mathbf{r}') \partial_j^{r'} \right]. \quad (103)$$

In accordance with our preceding discussion the \hat{X}_I have to be considered as creation operators in functional (transformed) Fock space. Thus the inverse of (103) has to be constructed in this space. However, it has to be observed that the resolution of the Gauß constraint is intended at the elimination of the longitudinal branch. Hence the representation of the resolvent has to be restricted to the longitudinal subspace of the \hat{X}^1 -Fock space. As on the other hand \hat{X}_I^1 can be decomposed into

$$\hat{X}_I^1 = \hat{X}_I^{1,\ell} + \hat{X}_I^{1,t} \quad (104)$$

this means that for the resolvent calculation $\hat{X}_I^{1,t}$ plays the role of an “external” parameter.

We observe that due to the commutativity of the new variables $\hat{X}_K^{1,t}, \hat{X}_K^{1,\ell}, Z_I^1$ the corresponding functional Fock space basis can be represented by

$$\mathcal{F}(h)^1 := \left\{ \left[\bigoplus_n \left(\hat{X}_K^{1,t} \right)^n |0\rangle_F^t \right] \otimes \left[\bigoplus_m \left(\hat{X}_K^{1,\ell} \right)^m |0\rangle_F^\ell \right] \otimes \left[\bigoplus_r \left(Z_I^1 \right)^r |0\rangle_F^f \right] \right\}, \quad (105)$$

where $|0\rangle_F^\ell$ is the Fock vacuum of the longitudinal branch, $|0\rangle_F^t$ that of the transversal branch, and $|0\rangle_F^f$ represents the fermionic functional Fock vacuum.

Proposition 4: A representation of $|\mathcal{G}\rangle^1$ in the new variables \hat{X}_K^1, Z_I^1 is given by

$$|\mathcal{G}\rangle^1 = \sum_{nm} \frac{i^n}{n!} \frac{i^m}{m!} g_{nm}(I_1 \dots I_n, K_1 \dots K_m) \cdot Z_{I_1}^1 \dots Z_{I_n}^1 \hat{X}_{K_1}^1 \dots \hat{X}_{K_m}^1 |0\rangle_F' \quad (106)$$

The representation (106) can be equivalently expressed by

$$|\mathcal{G}\rangle^1 = \sum_h \frac{i^h}{h!} |\mathcal{G}_h^t(K_1 \dots K_h)\rangle^{f,\ell} \hat{X}_{K_1}^{1,t} \dots \hat{X}_{K_h}^{1,t} |0\rangle_F^t, \quad (107)$$

or

$$|\mathcal{G}\rangle^1 = \sum_j \frac{i^j}{j!} |\mathcal{G}_j^\ell(K_1 \dots K_j)\rangle^{f,t} \hat{X}_{K_1}^{1,\ell} \dots \hat{X}_{K_j}^{1,\ell} |0\rangle_F^\ell, \quad (108)$$

where $|\mathcal{G}_h^t(K_1 \dots K_h)\rangle^{f,\ell}$ is a functional state in the fermion variables and the longitudinal branch, while $|\mathcal{G}_j^\ell(K_1 \dots K_j)\rangle^{f,t}$ is a functional state in the fermion variables and the transversal branch. These states are invariant under application of P_t or P_ℓ on $|\mathcal{G}_h^t\rangle$ or $|\mathcal{G}_j^\ell\rangle$, respectively.

Proof: As the transformed functional space is again a Fock space, equation (106) is the most general representation of $|\mathcal{G}\rangle^1$ in this space. Due to $|0\rangle_F' = |0\rangle_F^f \otimes |0\rangle_F^b$ we write (106) in the equivalent form

$$|\mathcal{G}\rangle^1 = \sum_m \frac{i^m}{m!} |\mathcal{G}_m(K_1 \dots K_m)\rangle^f \hat{X}_{K_1}^1 \dots \hat{X}_{K_m}^1 |0\rangle_F^b. \quad (109)$$

Using the properties of the projection operators $P_\alpha^\ell, P_\alpha^t$ which act on the coordinate K_α , we obtain

$$\begin{aligned} |\mathcal{G}\rangle^1 &= \sum_m \frac{i^m}{m!} |\mathcal{G}_m(K_1 \dots K_m)\rangle^f \\ &\quad \cdot \prod_{\alpha=1}^m (P_\alpha^\ell + P_\alpha^t)^2 \hat{X}_{K_1}^1 \dots \hat{X}_{K_m}^1 |0\rangle_F^b \\ &= \sum_m \frac{i^m}{m!} |\mathcal{G}_m(K_1 \dots K_m)\rangle^f \\ &\quad \cdot \prod_{\alpha=1}^m \left[(P_\alpha^\ell)^2 + (P_\alpha^t)^2 \right] \hat{X}_{K_1}^1 \dots \hat{X}_{K_m}^1 |0\rangle_F^b \end{aligned} \quad (110)$$

or equivalently

$$\begin{aligned} |\mathcal{G}\rangle^1 &= \sum_m \frac{i^m}{m!} \sum_{i_1 \dots i_m} |\mathcal{G}_m(K_1 \dots K_m)\rangle^f \\ &\quad \cdot (P_{i_1}^1)^2 \dots (P_{i_m}^m)^2 \hat{X}_{K_1}^1 \dots \hat{X}_{K_m}^1 |0\rangle_F^b \\ &= \sum_m \frac{i^m}{m!} \sum_{i_1 \dots i_m} (P_{i_1}^{i_1} \dots P_{i_m}^{i_m}) |\mathcal{G}_m(K_1 \dots K_m)\rangle^f \\ &\quad \cdot (P_{i_1}^{i_1}) \hat{X}_{K_1}^1 \dots (P_{i_m}^{i_m}) \hat{X}_{K_m}^1 |0\rangle_F^b \end{aligned} \quad (111)$$

with $i_\alpha = 1, 2 \equiv \ell, t$. In the second part of (111) the squares of the projection operators were resolved in

favor of a symmetric application on \hat{X}^1 and $|\mathcal{G}_m\rangle$. This step can be explicitly verified in coordinate space or by Fourier transformation. Due to (105) a rearrangement in the $(P_\alpha^i)\hat{X}_{K_\alpha}^1$ variables leads to (107) or (108), respectively. Furthermore, as $(P_\alpha^i)^2 = P_\alpha^i$ the invariance property can be verified. \diamond

These representations of $|\mathcal{G}\rangle^1$ allow a stepwise construction of the resolvent for (103) in the various subspaces. We write (102) in the form

$$\begin{aligned} & \int \left[\delta_{ac} \delta(\mathbf{r} - \mathbf{r}') + g \frac{1}{4\pi} f_{abc} C(\mathbf{r}, \mathbf{r}') \hat{X}_{bj}^{1,\ell}(\mathbf{r}') \partial_j^{r'} \right. \\ & \quad \left. + g \frac{1}{4\pi} f_{abc} C(\mathbf{r}, \mathbf{r}') \hat{X}_{bj}^{1,t}(\mathbf{r}') \partial_j^{r'} \right] \partial_c^1(\mathbf{r}') d^3 r' |\mathcal{G}\rangle^1 \\ & = -\frac{1}{4\pi} \int C(\mathbf{r}, \mathbf{r}') \hat{\rho}_a^1(\mathbf{r}') d^3 r' |\mathcal{G}\rangle^1 \end{aligned} \quad (112)$$

and replace it by the symbolic notation

$$(1 + \mathcal{O}_1 + \mathcal{O}_2) \partial |\mathcal{G}\rangle^1 = \mathcal{O}_3 |\mathcal{G}\rangle^1. \quad (113)$$

First we resolve this equation formally.

Proposition 5: With respect to $\partial |\mathcal{G}\rangle^1 \equiv \hat{\partial}_a^{1,\ell}(\mathbf{r}) |\mathcal{G}\rangle^1$ equation (113) is equivalent to equation

$$\partial |\mathcal{G}\rangle^1 = (1 + \mathcal{R}_2^t \mathcal{O}_1)^{-1} \mathcal{R}_2^t \mathcal{O}_3 |\mathcal{G}\rangle^1, \quad (114)$$

where \mathcal{R}_2^t is the right inverse of $(1 + \mathcal{O}_2)$ in the transversal Fock subspace of (105).

Proof: We first show that (113) follows from (114). Then we prove the reverse.

i) We multiply (114) by $(1 + \mathcal{R}_2^t \mathcal{O}_1)$ and obtain

$$(1 + \mathcal{R}_2^t \mathcal{O}_1) \partial |\mathcal{G}\rangle^1 = \mathcal{R}_2^t \mathcal{O}_3 |\mathcal{G}\rangle^1 \quad (115)$$

or

$$\partial |\mathcal{G}\rangle^1 = \mathcal{R}_2^t \mathcal{O}_3 |\mathcal{G}\rangle^1 - \mathcal{R}_2^t \mathcal{O}_1 \partial |\mathcal{G}\rangle^1. \quad (116)$$

Multiplication by $(1 + \mathcal{O}_2)$ yields

$$(1 + \mathcal{O}_2) \partial |\mathcal{G}\rangle^1 = \mathcal{O}_3 |\mathcal{G}\rangle^1 - \mathcal{O}_1 \partial |\mathcal{G}\rangle^1 \quad (117)$$

i.e. Equation (113).

ii) We generalize (113) to

$$(1 + \mathcal{O}_2 + \mathcal{O}_1) \partial |\mathcal{G}\rangle^1 = \mathcal{O}_3 |\mathcal{F}\rangle. \quad (118)$$

We apply the ansatz $\partial |\mathcal{G}\rangle^1 = X |\mathcal{F}\rangle$. Substitution into (118) gives

$$(1 + \mathcal{O}_2) X |\mathcal{F}\rangle = \mathcal{O}_3 |\mathcal{F}\rangle - \mathcal{O}_1 X |\mathcal{F}\rangle. \quad (119)$$

For $X = \mathcal{R}_2^t Y$ one obtains

$$Y |\mathcal{F}\rangle = \mathcal{O}_3 |\mathcal{F}\rangle - \mathcal{O}_1 \mathcal{R}_2^t Y |\mathcal{F}\rangle \quad (120)$$

or

$$(1 + \mathcal{O}_1 \mathcal{R}_2^t) Y |\mathcal{F}\rangle = \mathcal{O}_3 |\mathcal{F}\rangle. \quad (121)$$

For $Y = (1 + \mathcal{O}_1 \mathcal{R}_2^t)^{-1} \mathcal{O}_3$ Eq. (121) leads to the identity $\mathcal{O}_3 |\mathcal{F}\rangle = \mathcal{O}_3 |\mathcal{F}\rangle$. This means

$$\partial |\mathcal{G}\rangle^1 = \mathcal{R}_2^t (1 + \mathcal{O}_1 \mathcal{R}_2^t)^{-1} \mathcal{O}_3 |\mathcal{F}\rangle \quad (122)$$

fulfills (118). By definition \mathcal{O}_1 is an operator in the longitudinal subspace. Hence $[\mathcal{R}_2^t, \mathcal{O}_1]_- = 0$. Therefore $\mathcal{R}_2^t (1 + \mathcal{O}_1 \mathcal{R}_2^t)^{-1} = (1 + \mathcal{O}_1 \mathcal{R}_2^t)^{-1} \mathcal{R}_2^t$ and with $|\mathcal{F}\rangle \equiv |\mathcal{G}\rangle^1$ it follows (114). \diamond

After the formal resolution of (113) by right-inverses we directly construct these resolvents. For abbreviation we use the following notation for the transversal Fock states

$$\begin{matrix} b_1 & b_k \\ | \mathbf{z}_1 \dots \mathbf{z}_k \rangle^t := \hat{X}_{h_1 b_1}^{1,t}(\mathbf{z}_1) \dots \hat{X}_{h_k b_k}^{1,t}(\mathbf{z}_k) |0\rangle_F^t, & (123) \\ h_1 & h_k \end{matrix}$$

$${}^t \langle \begin{matrix} b_k & b_1 \\ \mathbf{z}_k \dots \mathbf{z}_1 \\ h_k & h_1 \end{matrix} | := \left(\begin{matrix} b_1 & b_k \\ | \mathbf{z}_1 \dots \mathbf{z}_k \rangle^t \\ h_1 & h_k \end{matrix} \right)^+. \quad (124)$$

Then due to (82) and (96) for these states the following scalarproducts result

$$\begin{aligned} & {}^t \langle \begin{matrix} d_1 & d_m & b_1 & b_k \\ \mathbf{u}_1 \dots \mathbf{u}_m & \mathbf{z}_1 \dots \mathbf{z}_k \\ k_1 & k_m & h_1 & h_k \end{matrix} | \begin{matrix} b_1 & b_k \\ \mathbf{z}_1 \dots \mathbf{z}_k \\ h_1 & h_k \end{matrix} \rangle^t \\ & = \delta_{mk} P \sum_{\lambda_1 \dots \lambda_m} D^t(\mathbf{u}_1 \quad \mathbf{z}_{\lambda_1}) \dots D^t(\mathbf{u}_m \quad \mathbf{z}_{\lambda_m}) \end{aligned} \quad (125)$$

with

$$D^t := \delta_{db} \left[\delta(\mathbf{u} - \mathbf{v}) \delta_{kh} - \partial_k^{\mathbf{u}} C(\mathbf{u}, \mathbf{v}) \partial_h^{\mathbf{v}} \right]. \quad (126)$$

Furthermore it is $P^t X_K^{1,t} = X_K^{1,t}$. By means of these formulas we can now express \mathcal{R}_2^t .

Proposition 6: The nonperturbative right-inverse of the integral kernel

$$K^t(\mathbf{r}, \mathbf{r}') := \left[\delta_{ac} \delta(\mathbf{r} - \mathbf{r}') + g \frac{1}{4\pi} f_{abc} C(\mathbf{r}, \mathbf{r}') \hat{X}_{bj}^{1,t}(\mathbf{r}') \partial_j^{r'} \right] \quad (127)$$

is given by the resolvent integral kernel

$$\mathcal{R}_2^t(\mathbf{r}, \mathbf{r}') := \sum_{k,n} \frac{1}{k!n!} \int R^{kn}(\mathbf{r}, \mathbf{z}_1 \dots \mathbf{z}_k, \mathbf{r}', \mathbf{r}_1 \dots \mathbf{r}_n) \left| \mathbf{z}_1 \dots \mathbf{z}_k \right\rangle^t \cdot {}^t \langle \mathbf{r}_1 \dots \mathbf{r}_n | \mathbf{d}^3 z_1 \dots \mathbf{d}^3 z_k \mathbf{d}^3 r_1 \dots \mathbf{d}^3 r_n \quad (128)$$

with

$$R^{kn} = 0 \quad \text{if } n \geq k+1, k=0, 1 \dots \infty$$

and the diagonal term ($k=0, 1 \dots \infty$)

$$R^{kk} \left(\begin{array}{cccc} a & b_1 & b_k & c \\ \mathbf{r}, \mathbf{z}_1 \dots \mathbf{z}_k, \mathbf{r}', \mathbf{r}_1 \dots \mathbf{r}_k & h_1 & h_k & \ell_1 \end{array} \begin{array}{cc} a_1 & a_k \\ \ell_1 & \ell_k \end{array} \right) = \delta_{ac} \delta(\mathbf{r} - \mathbf{r}') P \sum_{\lambda_1 \dots \lambda_k} D^t(\mathbf{z}_1 \mathbf{r}_{\lambda_1}) \dots D^t(\mathbf{z}_k \mathbf{r}_{\lambda_k}) \cdot \quad (129)$$

The nondiagonal terms for $n < k$ can be determined by the recurrence formula

$$R^{kn} \left(\begin{array}{cccc} a & b_1 & b_k & c \\ \mathbf{r}, \mathbf{z}_1 \dots \mathbf{z}_k, \mathbf{r}', \mathbf{r}_1 \dots \mathbf{r}_n & h_1 & h_k & \ell_1 \end{array} \begin{array}{cc} a_1 & a_n \\ \ell_1 & \ell_n \end{array} \right) = -g \frac{1}{4\pi} \sum_{s=1}^k f_{abs} C(\mathbf{r}, \mathbf{z}_s) \partial_{h_s}^{\mathbf{z}_s} R^{k-1n} \left(\begin{array}{cccc} b & b_1 & b_s & c \\ \mathbf{z}_s, \mathbf{z}_1 \dots \mathbf{z}_s \dots \mathbf{z}_k, \mathbf{r}', \mathbf{r}_1 \dots \mathbf{r}_n & h_1 & h_s & h_k \end{array} \begin{array}{cc} a_1 & a_n \\ \ell_1 & \ell_n \end{array} \right), \quad (130)$$

where the dot means omitting the corresponding coordinates.

Proof: We define \mathcal{R}_2^t by the postulate

$$\int K^t(\mathbf{r}, \mathbf{r}'') \mathcal{R}_2^t(\mathbf{r}'', \mathbf{r}') \mathbf{d}^3 r'' \stackrel{!}{=} \mathbb{I}^t(\mathbf{r}, \mathbf{r}') := \sum_n \frac{1}{n!} \delta(\mathbf{r} - \mathbf{r}') \delta_{ab} \int \left| \mathbf{y}_1 \dots \mathbf{y}_n \right\rangle^t \cdot {}^t \langle \mathbf{y}_1 \dots \mathbf{y}_n | \mathbf{d}^3 y_1 \dots \mathbf{d}^3 y_n. \quad (131)$$

Let $|\mathcal{F}(\mathbf{r})\rangle$ and $|\mathcal{D}(\mathbf{r})\rangle$ be two functional states in transversal Fock space of type (113). Then we study the inversion of the equation

$$\int K^t(\mathbf{r}, \mathbf{r}'') |\mathcal{F}(\mathbf{r}'')\rangle \mathbf{d}^3 r'' = |\mathcal{D}(\mathbf{r})\rangle. \quad (132)$$

We substitute the inversion

$$|\mathcal{F}(\mathbf{r}'')\rangle = \int \mathcal{R}_2^t(\mathbf{r}'', \mathbf{r}') |\mathcal{D}(\mathbf{r}')\rangle \mathbf{d}^3 r' \quad (133)$$

into (132) and obtain by (131)

$$\int \mathbb{I}^t(\mathbf{r}, \mathbf{r}') |\mathcal{D}(\mathbf{r}')\rangle \mathbf{d}^3 r' = |\mathcal{D}(\mathbf{r})\rangle. \quad (134)$$

Using (131) for the evaluation of the left-hand side of (134) and taking into account the invariance properties

$|\mathcal{D}(\mathbf{r})\rangle$ against transversal projections, we obtain

$|\mathbb{I}^t|\mathcal{D}\rangle \equiv |\mathcal{D}\rangle$, i.e., the condition (133) is the resolution of (132), i.e., if by direct construction the postulate (131) can be fulfilled, \mathcal{R}_2^t is the exact right inverse of K^t .

For the explicit construction of \mathcal{R}_2^t we substitute (127) and (128) into (131) and project (131) from

the left by ${}^t \langle \begin{smallmatrix} d_1 & d_m \\ \mathbf{u}_1 \dots \mathbf{u}_m \\ h_1 & h_m \end{smallmatrix} \mid$ and from the right by

$\begin{smallmatrix} e_1 & e_q \\ \mathbf{v}_1 \dots \mathbf{v}_q \\ r_1 & r_q \end{smallmatrix} \rangle^t$. Due to (125) all scalar products can be

exactly evaluated. For $m, q = 0, 1 \dots \infty$ this leads to a system of equations for the determination of R^{kn} . The exact resolution of this system can be formulated by the relations given above. The nonperturbative evaluation of this system is elementary. For brevity we do not explicitly reproduce the corresponding lengthy formulas. \diamond

To perform the next step we define in analogy to (123) the longitudinal branch by

$$\begin{smallmatrix} b_1 & b_k \\ \mid \mathbf{z}_1 \dots \mathbf{z}_k \\ h_1 & h_k \end{smallmatrix} \rangle^\ell := \hat{X}_{h_1 b_1}^{1,\ell}(\mathbf{z}_1) \dots \hat{X}_{h_k b_k}^{1,\ell}(\mathbf{z}_k) \mid 0 \rangle_F^\ell, \quad (135)$$

$$\begin{smallmatrix} b_k & b_1 \\ \mid \mathbf{z}_k \dots \mathbf{z}_1 \\ h_k & h_1 \end{smallmatrix} := \left(\begin{smallmatrix} b_1 & b_k \\ \mid \mathbf{z}_1 \dots \mathbf{z}_k \\ h_1 & h_k \end{smallmatrix} \rangle^\ell \right)^+ \quad (136)$$

which yields the scalar product

$$\begin{smallmatrix} d_1 & d_m & b_1 & b_k \\ \mid \mathbf{u}_1 \dots \mathbf{u}_m \mid \mathbf{z}_1 \dots \mathbf{z}_k \\ k_1 & k_m & h_1 & h_k \end{smallmatrix} \rangle^\ell \quad (137)$$

$$= \delta_{mk} P \sum_{\lambda_1 \dots \lambda_m} D^\ell(\mathbf{u}_1 \mid \mathbf{z}_{\lambda_1}) \dots D^\ell(\mathbf{u}_m \mid \mathbf{z}_{\lambda_m})$$

with

$$D^\ell := \delta_{db} \partial_k^u C(\mathbf{u}, \mathbf{v}) \partial_h^v. \quad (138)$$

These formulas can be used in the following theorem.

Proposition 7: The nonperturbative right inverse of the integral kernel

$$\hat{K} \begin{smallmatrix} b \\ a & c \end{smallmatrix} (\mathbf{r}, \mathbf{r}') := \left[\delta_{ac} \delta(\mathbf{r} - \mathbf{r}') + g \frac{1}{4\pi} \hat{C} \begin{smallmatrix} b \\ a & c \end{smallmatrix} (\mathbf{r}, \mathbf{r}') \hat{X}_{b_j}^{1,\ell}(\mathbf{r}') \partial_j^{r'} \right] \quad (139)$$

with

$$\hat{C} \begin{smallmatrix} b \\ a & c \end{smallmatrix} (\mathbf{r}, \mathbf{r}') := \int \mathcal{R}_2^t(\mathbf{r}, \mathbf{r}'') f_{dbc} C(\mathbf{r}'', \mathbf{r}') d^3 r'' \quad (140)$$

is given by

$$\hat{R} \begin{smallmatrix} b \\ a & c \end{smallmatrix} (\mathbf{r}, \mathbf{r}') := \sum_{k,n} \frac{1}{k!n!} \quad (141)$$

$$\cdot \int \hat{R}^{kn} \begin{smallmatrix} a & b_1 & b_k & c & a_1 & a_n & b_1 & b_k \\ h_1 & h_k & h_k & \ell_1 & \ell_n & h_1 & h_k \end{smallmatrix} (\mathbf{r}, \mathbf{z}_1 \dots \mathbf{z}_k, \mathbf{r}', \mathbf{r}_1 \dots \mathbf{r}_n) \mid \mathbf{z}_1 \dots \mathbf{z}_k \rangle^\ell$$

$$\cdot {}^\ell \langle \begin{smallmatrix} a_1 & a_n \\ \mathbf{r}_1 \dots \mathbf{r}_n \\ \ell_1 & \ell_n \end{smallmatrix} \mid d^3 z_1 \dots d^3 z_k d^3 r_1 \dots d^3 r_n$$

with

$$\hat{R}^{kn} = 0 \quad \text{if } n \geq k+1, k=0, 1 \dots \infty,$$

$$\hat{R}^{kk} \begin{smallmatrix} a & b_1 & b_k & c & a_1 & a_k \\ h_1 & h_k & h_k & \ell_1 & \ell_k \end{smallmatrix} (\mathbf{r}, \mathbf{z}_1 \dots \mathbf{z}_k, \mathbf{r}, \mathbf{r}_1 \dots \mathbf{r}_k) \quad (142)$$

$$= \delta_{ac} \delta(\mathbf{r} - \mathbf{r}') P \sum_{\lambda_1 \dots \lambda_k} D^\ell(\mathbf{z}_1 \mid \mathbf{r}_{\lambda_1}) \dots D^\ell(\mathbf{z}_k \mid \mathbf{r}_{\lambda_k})$$

and the recurrence formula

$$\hat{R}^{kn} \begin{smallmatrix} a & b_1 & b_k & c & a_1 & a_n \\ h_1 & h_k & h_k & \ell_1 & \ell_n \end{smallmatrix} (\mathbf{r}, \mathbf{z}_1 \dots \mathbf{z}_k, \mathbf{r}', \mathbf{r}_1 \dots \mathbf{r}_n) \quad (143)$$

$$= -g \frac{1}{4\pi} \sum_{s=1}^k \left[\hat{C} \begin{smallmatrix} b_s \\ a & b \end{smallmatrix} (\mathbf{r} \mid \mathbf{z}_s) \partial_{h_s}^{z_s} \hat{R}^{k-1,n} \right.$$

$$\cdot \left. \begin{smallmatrix} b & b_1 & b_s & b_k & c & a_1 & a_n \\ \mathbf{z}_s, \mathbf{z}_1 \dots \mathbf{z}_s & \dots & \mathbf{z}_k, \mathbf{r}', \mathbf{r}_1 \dots \mathbf{r}_n \\ h_1 & h_s & h_k & \ell_1 & \ell_n \end{smallmatrix} \right] .$$

Proof: In contrast to the integral kernel (127) with the ordinary c -number Coulomb potential $C(\mathbf{r}, \mathbf{r}')$, in the present case the integral kernel (139) contains the operator valued generalized Coulomb potential $\hat{C}(\mathbf{r}, \mathbf{r}')$. In spite of this generalization the proof runs along the same lines as that of Proposition 3. \diamond

By means of these propositions an exact nonperturbative and explicit expression of formula (114) can be

given which we will use in the following. We combine the results of the preceding propositions and obtain for the right inverse of the integral kernel

$$\mathcal{K}_{a \ c}(\mathbf{r}, \mathbf{r}') := \left[\delta_{ac} \delta(\mathbf{r} - \mathbf{r}') + g \frac{1}{4\pi} f_{abc} C(\mathbf{r}, \mathbf{r}') \hat{X}_{bj}^1(\mathbf{r}') \partial_j^{r'} \right] \quad (144)$$

the expression

$$\begin{aligned} \mathcal{R}_{a \ c}(\mathbf{r}, \mathbf{r}') &:= (1 + \mathcal{R}_2^t \mathcal{O}_1)^{-1} \mathcal{R}_2^t \\ &\equiv \int \hat{\mathcal{R}}_{a \ d}(\mathbf{r} \ \mathbf{r}'') \mathcal{R}_2^t(\mathbf{r}'' \ \mathbf{r}') d^3 r'' \end{aligned} \quad (145)$$

and thus Eq. (114) can be written in the form

$$\begin{aligned} \partial_a^1(\mathbf{r})|\mathcal{G}\rangle^1 &= -\frac{1}{4\pi} \int \mathcal{R}_{a \ c}(\mathbf{r} \ \mathbf{r}') C(\mathbf{r}' \ \mathbf{r}'') \\ &\quad \cdot \hat{\rho}_c^1(\mathbf{r}'') d^3 r' d^3 r'' |\mathcal{G}\rangle^1. \end{aligned} \quad (146)$$

In the next step we investigate the longitudinal color electric energy in (99) by means of this formula.

Proposition 8: The color electric energy in functional version applied to physical functional states $|\mathcal{G}\rangle^1$ is given by

$$\begin{aligned} -\frac{1}{2} \int \partial_j^1(\mathbf{r}) \partial_j^1(\mathbf{r}) d^3 r |\mathcal{G}\rangle^1 &= -\frac{1}{2} \int d^3 r d^3 r' d^3 y d^3 z \\ &\quad \cdot \delta(\mathbf{r} - \mathbf{z}) \delta_{ad} \partial_j^1 \partial_j^1 \left\{ \frac{1}{(4\pi)^2} \left[\mathcal{R}_{a \ f}(\mathbf{r}, \mathbf{y}) C(\mathbf{y}, \mathbf{r}') \hat{\rho}_f^1(\mathbf{r}') \right. \right. \\ &\quad \cdot \mathcal{R}_{d \ e}(\mathbf{z}, \mathbf{x}) C(\mathbf{x}, \mathbf{r}'') \hat{\rho}_e^1(\mathbf{r}'') + \mathcal{R}_{a \ f}(\mathbf{r}, \mathbf{y}) g f_{fbc} C(\mathbf{y}, \mathbf{r}') \delta_{bd} \\ &\quad \cdot \hat{P}_j(\mathbf{r}', \mathbf{z}) \partial_j^1 \mathcal{R}_{c \ e}(\mathbf{r}', \mathbf{x}) C(\mathbf{x}, \mathbf{r}'') \hat{\rho}_e^1(\mathbf{r}'') \left. \right] d^3 x d^3 r'' \\ &\quad \left. - \frac{1}{4\pi} \mathcal{R}_{a \ f}(\mathbf{r}, \mathbf{y}) C(\mathbf{y}, \mathbf{r}') f_{fbc} \delta_{bd} \hat{P}_j(\mathbf{r}', \mathbf{z}) \hat{\rho}_{cj}^{1,t}(\mathbf{r}') \right\} |\mathcal{G}\rangle^1 \end{aligned} \quad (147)$$

with

$$\delta_{ab} \hat{P}_j(\mathbf{r}', \mathbf{z}) := \left[\partial_a^1(\mathbf{r}'), \hat{X}_{bj}^1(\mathbf{z}) \right]_- . \quad (148)$$

Proof: We apply $\partial_a^1(\mathbf{z})$ from the left to the Gauß law (112) and commute it with the functional Gauß operator, i.e. generator, so that $\partial_a^1(\mathbf{z})$ directly acts on $|\mathcal{G}\rangle^1$. Successive application of the resolvent (145) and subsequent contraction by $\delta(\mathbf{r} - \mathbf{z}) \delta_{ad}$ with integration over \mathbf{r} and \mathbf{z} leads to (147). \diamond

Obviously the first term on the right-hand side of (147) plays the role of the nonabelian color electric Coulomb field energy, while the following terms stem from the commutators, i.e. they add to the color “Coulomb” forces specific quantum corrections. In this way we have realized our intention to exactly resolve the Gauß constraint and to incorporate it into the quantum Hamilton operator. However, as can also be seen, in (147) the functional representatives of the A^ℓ -fields are still present. Thus, in a last step we have to eliminate these redundant variables.

7. Effective functional energy equations

We continue the discussion of the $i = 1$ case, because already at this stage the effective color Coulomb forces can be identified.

Proposition 9: The effective functional Hamiltonian $\mathcal{H}_1^{\text{eff}}$ for the $i = 1$ case resulting from the elimination of the Gauß constraints and the longitudinal vector potentials is given by

$$\begin{aligned} \mathcal{H}_1^{\text{eff}} &:= -\frac{1}{2} \int \hat{\partial}_{ja}^{1,t}(\mathbf{r}) \hat{\partial}_{ja}^{1,t}(\mathbf{r}) d^3 r \\ &\quad + \frac{1}{2} \int \left[\epsilon_{jhl} \partial_h \hat{X}_{la}^{1,t}(\mathbf{r}) + \frac{1}{2} \epsilon_{jhl} g f_{abc} \hat{X}_{hb}^{1,t}(\mathbf{r}) \hat{X}_{lc}^{1,t}(\mathbf{r}) \right] \\ &\quad \cdot \left[\epsilon_{jhl} \partial_h \hat{X}_{la}^{1,t}(\mathbf{r}) + \frac{1}{2} \epsilon_{jhl} g f_{ab'c'} \hat{X}_{h'b'}^{1,t}(\mathbf{r}) \hat{X}_{l'c'}^{1,t}(\mathbf{r}) \right] d^3 r \\ &\quad - \int \left[Z_{\alpha A}^1(\mathbf{r}) (i\alpha^k \partial_k - \beta m)_{\alpha\beta} \partial_{\beta A}^1(\mathbf{r}) \right. \\ &\quad \left. - g \hat{X}_{ja}^{1,t}(\mathbf{r}) Z_{\alpha A}^1(\mathbf{r}) \alpha_{\alpha\beta}^j L_{AB}^a \partial_{\beta B}^1(\mathbf{r}) \right] d^3 r \\ &\quad - \frac{1}{2} \int d^3 r d^3 r' d^3 y d^3 z \delta(\mathbf{r} - \mathbf{z}) \delta_{ad} \partial_j^1 \partial_j^1 \left\{ \frac{1}{(4\pi)^2} \right. \\ &\quad \cdot \left[\mathcal{R}_{a \ f}^t(\mathbf{r}, \mathbf{y}) C(\mathbf{y}, \mathbf{r}') \hat{\rho}_f^0(\mathbf{r}') \mathcal{R}_2^t(\mathbf{z}, \mathbf{x}) C(\mathbf{x}, \mathbf{r}'') \hat{\rho}_e^0(\mathbf{r}'') \right. \\ &\quad \left. + \mathcal{R}_{a \ f}^t(\mathbf{r}, \mathbf{y}) g f_{fbc} C(\mathbf{y}, \mathbf{r}') \delta_{bd} \hat{P}_j(\mathbf{r}', \mathbf{z}) \partial_j^1 \mathcal{R}_{c \ e}^t(\mathbf{r}', \mathbf{x}) \right. \end{aligned} \quad (149)$$

$$\cdot C(\mathbf{x}, \mathbf{r}'') \hat{\rho}_e^0(\mathbf{r}'') \Big] d^3x d^3r'' \\ - \frac{1}{4\pi} \mathcal{R}_2^t(\mathbf{r}, \mathbf{y}) \Big|_{a \ f} C(\mathbf{y}, \mathbf{r}') f_{fbc} \delta_{bd} \hat{P}_j(\mathbf{r}', \mathbf{z}) \partial_{cj}^{1,t}(\mathbf{r}') \Big\}$$

with \mathcal{R}_2^t defined by (128).

Proof: We use for the functional color electric energy the division (98) and substitute expression (147) into the longitudinal part of (98). Furthermore we substitute $\hat{X}_K^1 = \hat{X}_K^{1,t} + \hat{X}_K^{1,\ell}$ and (98) into the functional Hamiltonian (76). By means of these substitutions we obtain from (76) and (99)

$$\mathcal{H}_1 := \text{r.h.s.}(147) + h_1 \left(\hat{\partial}^{1,t}, \hat{X}^{1,\ell}, \hat{X}^{1,t}, Z^1, \partial^1 \right) \quad (150)$$

By comparison of (150) with (76) and (99) it follows that h_1 admits a power series expansion in $\hat{X}^{1,\ell}$ which we write symbolically as

$$h_1 = \sum_{\nu=0}^4 h_1^\nu \left(\hat{\partial}^{1,t}, \hat{X}^{1,t}, Z^1, \partial^1 \right) (\hat{X}^{1,\ell})^\nu \quad (151)$$

with highest power four of $\hat{X}_K^{1,\ell}$. In the same way we obtain, due to (101),

$$\hat{\rho} = \hat{\rho}^0 + \hat{\rho}^1 \hat{X}^{1,\ell} \quad (152)$$

a symbolic expansion of $\hat{\rho}_a(\mathbf{r})$. We now apply \mathcal{H}_1 to $|\mathcal{G}\rangle^1$ and again expand $|\mathcal{G}\rangle^1$ with respect to the longitudinal Fock subspace. In symbolic notation this leads to

$$|\mathcal{G}\rangle^1 = \sum_{\kappa} (\hat{X}^{1,\ell})^\kappa |0\rangle_F^\ell \otimes |\mathcal{G}_\kappa\rangle^1, \quad (153)$$

where the set of states $\{|\mathcal{G}_\kappa\rangle^1\}$ contains all other degrees of freedom of the system. In the same symbolic way we represent $\hat{\mathcal{R}}(\mathbf{r}, \mathbf{r}')$ in the longitudinal Fock subspace by

$$\mathcal{R} \equiv \sum_{\lambda\chi} (\hat{X}^{1,\ell})^\lambda |0\rangle_F^\ell R_{\lambda\chi} |0\rangle_F^\ell (\hat{\partial}^{1,\ell})^\chi, \quad (154)$$

where the coefficient functionals $\{R_{\lambda\chi}\}$ are operators in the functional subspace of all other degrees of freedom, i.e., the complement of the longitudinal

subspace with respect to the whole functional Fock space of the system.

We now consider the equation

$${}^\ell_F \langle 0 | \mathcal{H}_1 | \mathcal{G} \rangle^1 = E_1 {}^\ell_F \langle 0 | \mathcal{G} \rangle^1 \quad (155)$$

and evaluate the various terms separately. In doing so we concentrate only on the effects of the projection suppressing all other details. We characterize equations in this symbolic calculation by the sign $\stackrel{\text{symb}}{=}$. We observe $R_{\lambda\chi} \equiv 0$ for $\chi > \lambda$ and obtain for the first term on the right-hand side of (147) the symbolic expression

$$\begin{aligned} & {}^\ell_F \langle 0 | (1. \text{ term r.h.s. (147)}) | \mathcal{G} \rangle^1 \\ & \stackrel{\text{symb}}{=} R_{00} {}^\ell_F \langle 0 | (\hat{\rho}^0 + \hat{\rho}^1 \hat{X}^{1,\ell}) (\hat{X}^{1,\ell})^{\lambda'} | 0 \rangle_F^\ell \\ & \quad \cdot R_{\lambda'\kappa'} {}^\ell_F \langle 0 | (\hat{\partial}^{1,\ell})^{\chi'} (\hat{\rho}^0 + \hat{\rho}^1 \hat{X}^{1,\ell}) \\ & \quad \cdot (\hat{X}^{1,\ell})^\kappa | 0 \rangle_F^\ell |\mathcal{G}_\kappa\rangle^1 \\ & \stackrel{\text{symb}}{=} R_{00} \hat{\rho}^0 R_{00} \hat{\rho}^0 |\mathcal{G}_0\rangle^1. \end{aligned} \quad (156)$$

For the second and the third term on the right-hand side of (147) one gets

$$\begin{aligned} & {}^\ell_F \langle 0 | (2. \text{ and } 3. \text{ term r.h.s. (147)}) | \mathcal{G} \rangle^1 \\ & \stackrel{\text{symb}}{=} R_{00} {}^\ell_F \langle 0 | (\hat{X}^{1,\ell})^{\lambda'} | 0 \rangle_F^\ell R_{\lambda'\chi'} {}^\ell_F \langle 0 | (\hat{\partial}^{1,\ell})^{\chi'} \\ & \quad \cdot (\hat{\rho}^0 + \hat{\rho}^1 \hat{X}^{1,\ell}) (\hat{X}^{1,\ell})^\kappa | 0 \rangle_F^\ell |\mathcal{G}_\kappa\rangle^1 \\ & \quad - R_{00} {}^\ell_F \langle 0 | (\hat{X}^{1,\ell})^\kappa | 0 \rangle_F^\ell |\mathcal{G}_\kappa\rangle^1 \\ & \stackrel{\text{symb}}{=} R_{00} R_{00} \hat{\rho}^0 |\mathcal{G}_0\rangle^1 - R_{00} |\mathcal{G}\rangle^1. \end{aligned} \quad (157)$$

The term (151) yields

$$\begin{aligned} & {}^\ell_F \langle 0 | h_1 | \mathcal{G} \rangle^1 = {}^\ell_F \langle 0 | h_1^\nu (\hat{X}^{1,\ell})^\nu (\hat{X}^{1,\ell})^\kappa | 0 \rangle_F^\ell |\mathcal{G}_\kappa\rangle^1 \\ & = h_1^0 \left(\hat{\rho}^{1,t}, \hat{X}^{1,t}, Z^1, \partial^1 \right) |\mathcal{G}_0\rangle^1. \end{aligned} \quad (158)$$

If we collect all terms the following expression results for equation (155)

$$\begin{aligned} & {}^\ell_F \langle 0 | \mathcal{H}_1 | \mathcal{G} \rangle^1 \stackrel{\text{symb}}{=} \left[R_{00} \hat{\rho}^0 R_{00} \hat{\rho}^0 + R_{00} R_{00} \hat{\rho}^0 \right. \\ & \quad \left. - R_{00} + h_1^0 \right] |\mathcal{G}_0\rangle^1 = E_1 |\mathcal{G}_0\rangle^1. \end{aligned} \quad (159)$$

The term R_{00} is the expectation value of \mathcal{R} with respect to the longitudinal Fock vacuum, i.e., we have the definition

$$R_{00} := \int_{\mathbb{F}} \langle 0 | \mathcal{R} | 0 \rangle_{\mathbb{F}}^{\ell} \quad (160)$$

$$:= \int_{\mathbb{F}} \langle 0 | \hat{\mathcal{R}}(\mathbf{r}, \mathbf{r}'') | 0 \rangle_{\mathbb{F}}^{\ell} \otimes \mathcal{R}_2^t(\mathbf{r}', \mathbf{r}') \frac{d^3 r''}{d^3 c}.$$

From (142) it follows

$$\int_{\mathbb{F}} \langle 0 | \hat{\mathcal{R}}(\mathbf{r}, \mathbf{r}'') | 0 \rangle_{\mathbb{F}}^{\ell} = \delta_{ad} \delta(\mathbf{r} - \mathbf{r}'') \quad (161)$$

and hence $R_{00} \equiv \mathcal{R}_2^t$. If we substitute this value into (159) and retranslate (159) from the symbolic notation into the full notation we obtain the effective functional Hamiltonian (149). \diamond

From this result it follows immediately that only the term $|\mathcal{G}_0\rangle^1$ of (153) is needed for the eigenvalue calculation of (149). This can be formulated as follows:

Proposition 10: With (150), (151) and (159) the energy eigenstate of $\mathcal{H}_1^{\text{eff}}$ is given by $\int_{\mathbb{F}} \langle 0 | \mathcal{G} \rangle^1 \equiv |\mathcal{G}_0\rangle^1$.

This means: The longitudinal branch of the non-abelian vector potential is not needed for the calculation of the energy eigenvalue.

The treatment of the $i = 2$ case was already outlined in section 5. In order to obtain quantitative analytic expressions we discuss this treatment in more detail. We start with equations (92), which due to (82) can be written in the form

$$\int \left\{ -\frac{1}{2} \hat{\partial}_{ja}^2(\mathbf{r}) \hat{\partial}_{ja}^2(\mathbf{r}) \right. \quad (162)$$

$$+ \frac{1}{2} \left[\epsilon_{jhl} \partial_h \hat{X}_{\ell a}^2(\mathbf{r}) + \frac{1}{2} \epsilon_{jhl} g f_{abc} \hat{X}_{hb}^2(\mathbf{r}) \hat{X}_{\ell c}^2(\mathbf{r}) \right]$$

$$\cdot \left[\epsilon_{jh'l'} \partial_{h'} \hat{X}_{\ell'a}^2(\mathbf{r}) + \frac{1}{2} \epsilon_{jh'l'} g f_{ab'c'} \hat{X}_{h'b'}^2(\mathbf{r}) \hat{X}_{\ell'c'}^2(\mathbf{r}) \right]$$

$$- g \hat{X}_{ja}^2(\mathbf{r}) Z_{\alpha A}^2(\mathbf{r}) \alpha_{\alpha\beta}^j L_{AB}^a \partial_{\beta A}^2(\mathbf{r})$$

$$+ Z_{\alpha A}^2(\mathbf{r}) (i\alpha^k \partial_k - \beta m)_{\alpha\beta} \partial_{\beta A}^2(\mathbf{r}) \left. \right\} d^3 r |S\rangle^2 = E |S\rangle^2$$

and

$$\left[i \partial_k \hat{\partial}_k^2 + i g f_{abc} \hat{X}_{jc}^2(\mathbf{r}) \hat{\partial}_{jb}^2(\mathbf{r}) + \rho_a^2(\mathbf{r})^+ \right] |S\rangle^2 = 0. \quad (163)$$

Comparison of (162) and (163) with (76) and (79) shows the complete structural analogy of the case $i = 2$ to the case $i = 1$. Hence we can completely take over the treatment of the case $i = 1$ to the case $i = 2$. This in particular means that propositions 5-10 can be applied to the resolution of (162) and (163). The result is an effective energy equation for $|S_0\rangle^2$

$$\mathcal{H}_2^{\text{eff}+} |S_0\rangle^2 = E_2 |S_0\rangle^2, \quad (164)$$

where $|S_0\rangle^2$ is an energy eigenstate in functional space independent of the longitudinal branch, and the functional Hamiltonian $\mathcal{H}_2^{\text{eff}+}$ is given by the definition

$$\mathcal{H}_2^{\text{eff}+} := -\frac{1}{2} \int \hat{\partial}_{ja}^{2,t}(\mathbf{r}) \hat{\partial}_{ja}^{2,t}(\mathbf{r}) d^3 r \quad (165)$$

$$+ \frac{1}{2} \int \left[\epsilon_{jhl} \partial_h \hat{X}_{\ell a}^{2,t}(\mathbf{r}) + \frac{1}{2} \epsilon_{jhl} g f_{abc} \hat{X}_{hb}^{2,t}(\mathbf{r}) \hat{X}_{\ell c}^{2,t}(\mathbf{r}) \right]$$

$$\cdot \left[\epsilon_{jh'l'} \partial_{h'} \hat{X}_{\ell'a}^{2,t}(\mathbf{r}) + \frac{1}{2} \epsilon_{jh'l'} g f_{ab'c'} \hat{X}_{h'b'}^{2,t}(\mathbf{r}) \hat{X}_{\ell'c'}^{2,t}(\mathbf{r}) \right] d^3 r$$

$$+ \int \left[Z_{\alpha A}^2(\mathbf{r}) (i\alpha^k \partial_k - \beta m)_{\alpha\beta} \partial_{\beta A}^2(\mathbf{r}) \right.$$

$$\left. - g \hat{X}_{ja}^{2,t}(\mathbf{r}) Z_{\alpha A}^2(\mathbf{r}) \alpha_{\alpha\beta}^j L_{AB}^a \partial_{\beta B}^2(\mathbf{r}) \right] d^3 r$$

$$- \frac{1}{2} \int d^3 r d^3 r' d^3 y d^3 z \delta(\mathbf{r} - \mathbf{z}) \delta_{ad} \partial_h^{\mathbf{r}} \partial_h^{\mathbf{z}} \left\{ \frac{1}{(4\pi)^2} \right.$$

$$\cdot \left[\hat{\mathcal{R}}_2^t(\mathbf{r}, \mathbf{y}) \frac{C(\mathbf{y}, \mathbf{r}') \hat{\rho}_{\mathbb{F}}^0(\mathbf{r}') \hat{\mathcal{R}}_2^t(\mathbf{z}, \mathbf{x})}{a \quad f \quad d \quad e} C(\mathbf{x}, \mathbf{r}'') \hat{\rho}_e^0(\mathbf{r}'') \right.$$

$$+ \hat{\mathcal{R}}_2^t(\mathbf{r}, \mathbf{y}) \frac{g f_{fbc} C(\mathbf{y}, \mathbf{r}') \delta_{bd} \hat{P}_j(\mathbf{r}', \mathbf{z}) \partial_j^{\mathbf{r}''} \hat{\mathcal{R}}_2^t(\mathbf{r}', \mathbf{x})}{a \quad f \quad c \quad e}$$

$$\cdot C(\mathbf{x}, \mathbf{r}'') \hat{\rho}_e^0(\mathbf{r}'') \left. \right] d^3 x d^3 r''$$

$$\left. - \frac{1}{4\pi} \hat{\mathcal{R}}_2^t(\mathbf{r}, \mathbf{y}) \frac{C(\mathbf{y}, \mathbf{r}') f_{fbc} \delta_{bd} \hat{P}_j(\mathbf{r}', \mathbf{z}) \partial_{ej}^{1,t}(\mathbf{r}')}{a \quad f} \right\},$$

where $\hat{\mathcal{R}}_2^t$ and $\hat{\rho}_a^0$ are given by \mathcal{R}_2^t and ρ_a^0 if the $i = 1$ variables are replaced by $i = 2$ variables. By hermitian conjugation (164) goes over into

$${}^2\langle \mathcal{S}_0 | \mathcal{H}_2^{\text{eff}} = {}^2\langle \mathcal{S}_0 | E_2. \quad (166)$$

We now assume that this equation also has a right-hand side solution

$$\mathcal{H}_2^{\text{eff}} |\mathcal{G}_0\rangle^2 = E_2 |\mathcal{G}_0\rangle^2 \quad (167)$$

and combine this equation with

$$\mathcal{H}_1^{\text{eff}} |\mathcal{G}_0\rangle^1 = E_1 |\mathcal{G}_0\rangle^1 \quad (168)$$

resulting from proposition 7. With $|\mathcal{G}_0\rangle := |\mathcal{G}_0\rangle^1 \otimes |\mathcal{G}_0\rangle^2$ we then obtain

$$(\mathcal{H}_1^{\text{eff}} - \mathcal{H}_2^{\text{eff}}) |\mathcal{G}_0\rangle = \Delta E |\mathcal{G}_0\rangle \quad (169)$$

which is the genuine eigenvalue equation of quantum chromodynamics in temporal gauge where all redundant variables are removed. In accordance with its derivation equation (169) is formulated in the transformed new variables. Due to (167), (168) this means that with respect to (169) this equation can only be solved in a finite volume in order to avoid infinite energies E_1 and E_2 . Only in the original variables the transition to infinite volume can be performed. Hence we have to express (169) in these old variables in analogy to quantum electrodynamics.

However, in contrast to quantum electrodynamics the effective Coulomb forces in (149) and (165) are rather complicated and need an additional detailed evaluation of the general formulas given in (149) and (165). But this would go beyond the present investigation. Therefore, in a first step we take into account only the lowest order of \mathcal{R}_2^t . It is given by

$$\mathcal{R}_2^{t,0} := \delta_{ac} |0\rangle_F' \delta(\mathbf{r} - \mathbf{r}') {}_F^t \langle 0|. \quad (170)$$

Substitution of $\mathcal{R}_2^{t,0}$ and $\hat{\mathcal{R}}_2^{t,0}$ into (149) and (165) then leads to the following explicit form of equation (169)

$$\begin{aligned} & \left\{ -\frac{1}{2} \int \hat{\partial}_{ja}^{1,t}(\mathbf{r}) \hat{\partial}_{ja}^{1,t}(\mathbf{r}) d^3r + \frac{1}{2} \int \hat{X}_{ja}^{2,t}(\mathbf{r}) \hat{X}_{ja}^{2,t}(\mathbf{r}) d^3r \right. \\ & + \frac{1}{2} \int \left[\epsilon_{jht} \partial_h^{\mathbf{r}} \hat{X}_{\ell a}^{1,t}(\mathbf{r}) + \frac{1}{2} \epsilon_{jht} g f_{abc} \hat{X}_{hb}^{1,t}(\mathbf{r}) \hat{X}_{\ell c}^{1,t}(\mathbf{r}) \right] \\ & \cdot \left[\epsilon_{jh'\ell'} \partial_{h'}^{\mathbf{r}} \hat{X}_{\ell'a}^{1,t}(\mathbf{r}) + \frac{1}{2} \epsilon_{jh'\ell'} g f_{ab'c'} \hat{X}_{h'b'}^{1,t}(\mathbf{r}) \hat{X}_{\ell'c'}^{1,t}(\mathbf{r}) \right] d^3r \end{aligned}$$

$$\begin{aligned} & - \frac{1}{2} \int \left[\epsilon_{jht} \partial_h^{\mathbf{r}} \hat{\partial}_{\ell a}^{2,t}(\mathbf{r}) + \frac{1}{2} \epsilon_{jht} g f_{abc} \hat{\partial}_{hb}^{2,t}(\mathbf{r}) \hat{\partial}_{\ell c}^{2,t}(\mathbf{r}) \right] \\ & \cdot \left[\epsilon_{jh'\ell'} \partial_{h'}^{\mathbf{r}} \hat{\partial}_{\ell'a}^{2,t}(\mathbf{r}) + \frac{1}{2} \epsilon_{jh'\ell'} g f_{ab'c'} \hat{\partial}_{h'b'}^{2,t}(\mathbf{r}) \hat{\partial}_{\ell'c'}^{2,t}(\mathbf{r}) \right] d^3r \\ & - \int \left[Z_{\alpha A}^1(\mathbf{r}) (i\alpha^k \partial_k - \beta m)_{\alpha\beta} \partial_{\beta A}^1(\mathbf{r}) \right. \\ & \quad \left. - g \hat{X}_{ja}^{1,t}(\mathbf{r}) Z_{\alpha A}^1(\mathbf{r}) \alpha_{\alpha\beta}^j L_{AB}^a \partial_{\beta B}^1(\mathbf{r}) \right] d^3r \\ & - \int \left[Z_{\alpha A}^2(\mathbf{r}) (i\alpha^k \partial_k - \beta m)_{\alpha\beta} \partial_{\beta A}^2(\mathbf{r}) \right. \\ & \quad \left. - g \hat{\partial}_{ja}^{2,t}(\mathbf{r}) Z_{\alpha A}^2(\mathbf{r}) \alpha_{\alpha\beta}^j L_{AB}^a \partial_{\beta B}^2(\mathbf{r}) \right] d^3r \\ & + \frac{g^2}{8\pi} \int [\rho_a^1(\mathbf{r}) C(\mathbf{r}, \mathbf{r}') \rho_a^1(\mathbf{r}') - \rho_a^2(\mathbf{r}) C(\mathbf{r}, \mathbf{r}') \rho_a^2(\mathbf{r}')] \\ & \cdot d^3r d^3r' |0\rangle_F' {}_F^t \langle 0| \} |\mathcal{G}_0\rangle = \Delta E |\mathcal{G}_0\rangle, \end{aligned} \quad (171)$$

where due to (170) the last two terms in (149) and (165) vanish. In addition the field parts of $\hat{\rho}^{10}$ and $\hat{\rho}^{20}$ drop out; thus only ρ_a^1 and ρ_a^2 remain.

So we have obtained in the lowest order of \mathcal{R}_2^t the color Coulomb forces between quarks. But a comparison with the Coulomb forces in quantum electrodynamics which are contained in I (123) shows that even in this lowest order the color Coulomb forces deviate by the projector $|0\rangle_F' {}_F^t \langle 0|$ from the ordinary ones. This is a clear indication for the nonperturbative treatment of this problem.

The necessary retransformation of (171) to the original functional variables runs along the same lines as in quantum electrodynamics. Hence for details we refer to I and give the final result directly. We obtain the following equation from (171)

$$\begin{aligned} & \left\{ i \int b_{ja}^{E,t}(\mathbf{r}) \Delta \mathbf{r} \partial_{ja}^{A,t}(\mathbf{r}) d^3r + i \int b_{ja}^{A,t}(\mathbf{r}) \partial_{ja}^{E,t}(\mathbf{r}) d^3r \right. \\ & + ig \int b_{ja}^{E,t}(\mathbf{r}) f_{abc} \left[2 \partial_{kb}^{A,t}(\mathbf{r}) \partial_{jc}^{A,t}(\mathbf{r}) \right. \\ & \quad \left. - \partial_{kb}^{A,t}(\mathbf{r}) \partial_j^{\mathbf{r}} \partial_{kc}^{A,t}(\mathbf{r}) - \frac{1}{6} b_{kb}^{E,t}(\mathbf{r}) \partial_k^{\mathbf{r}} b_{jc}^{E,t}(\mathbf{r}) \right. \\ & \quad \left. + \frac{1}{12} b_{kb}^{E,t}(\mathbf{r}) \partial_j^{\mathbf{r}} b_{kc}^{E,t}(\mathbf{r}) \right] d^3r \\ & - ig^2 \int b_{ja}^{E,t}(\mathbf{r}) f_{abc} f_{cde} \left[\partial_{kb}^{A,t}(\mathbf{r}) \partial_{kd}^{A,t}(\mathbf{r}) \partial_{je}^{A,t}(\mathbf{r}) \right. \\ & \quad \left. - \frac{1}{4} b_{kb}^{E,t}(\mathbf{r}) b_{kd}^{E,t}(\mathbf{r}) b_{je}^{A,t}(\mathbf{r}) \right] d^3r \end{aligned}$$

$$\begin{aligned}
& - \int j_{\alpha\lambda A}(\mathbf{r})(i\alpha^k \partial_k - \beta m)_{\alpha\beta} \partial_{\beta\lambda A}(\mathbf{r}) d^3r \\
& - ig \int \partial_{ja}^{A,t}(\mathbf{r}) \left[j_{\alpha\lambda A}(\mathbf{r}) \alpha_{\alpha\beta}^j \sigma_{\lambda\lambda'}^3 L_{AB}^a \partial_{\beta\lambda' B}(\mathbf{r}) \right] d^3r \\
& - ig \frac{1}{2} \int b_{ja}^{E,t}(\mathbf{r}) \left[\partial_{\kappa\lambda A}(\mathbf{r}) (C\gamma^j)_{\kappa\beta} \sigma_{\lambda\lambda'}^2 L_{AB}^a \partial_{\beta\lambda' B}(\mathbf{r}) \right. \\
& \quad \left. - \frac{1}{4} j_{\alpha\lambda A}(\mathbf{r}) (\gamma^j C)_{\alpha\beta} \sigma_{\lambda\lambda'}^2 L_{AB}^a j_{\beta\lambda' B}(\mathbf{r}) \right] d^3r \\
& + \frac{g^2}{8\pi} \int j_{\alpha\lambda A}(\mathbf{r}) \left[(C\gamma^0)_{\alpha_1\alpha_2} \delta_{\alpha_3\alpha} (\hat{L}^a)_{A_1 A_2}^{\lambda_1 \lambda_2} \right. \\
& \quad \cdot (\hat{L}^a)_{\lambda_3 \lambda'}^{\lambda_3 \lambda'} \sigma_{\lambda' \lambda}^1 C(\mathbf{r}, \mathbf{r}_1) \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta(\mathbf{r} - \mathbf{r}_3) \Big]_{\text{as}(123)} \\
& \quad \cdot \left[\partial_{\alpha_1 \lambda_1 A_1}(\mathbf{r}_1) \partial_{\alpha_2 \lambda_2 A_2}(\mathbf{r}_2) \partial_{\alpha_3 \lambda_3 A_3}(\mathbf{r}_3) \right. \\
& \quad \left. - \frac{1}{4\pi} (C\gamma^0)_{\alpha_1 \alpha'_1} \sigma_{\lambda_1 \lambda'_1}^1 (C\gamma^0)_{\alpha_2 \alpha'_2} \sigma_{\lambda_2 \lambda'_2}^2 j_{\alpha'_1 \lambda'_1 A_1}(\mathbf{r}_1) \right. \\
& \quad \cdot j_{\alpha'_2 \lambda'_2 A_2}(\mathbf{r}_2) \partial_{\alpha_3 \lambda_3 A_3}(\mathbf{r}_3) \Big] d^3r d^3r_1 d^3r_2 d^3r_3 \\
& \quad \cdot |0\rangle_F^{t'} \langle 0| \Big\} |\mathcal{G}_0\rangle = \Delta E |\mathcal{G}_0\rangle. \tag{172}
\end{aligned}$$

So far we have treated the algebraic formalism for quantum chromodynamics and the elimination of constraints without any reference to special representations which include the introduction of inner products and the emergence of representation-dependent effective functional Hamiltonians. In particular the latter are the only data obtainable for the formation of a time evolution, which then is representation-dependent and in this precise sense effective. It should be emphasized that this fact is a radical departure from the common treatment of quantum fields by perturbation theory, enforced by the algebraic concepts of representations of abstract operator algebras. This was already discussed in the introduction of I. How to introduce such representations and so forth into the representation-free dynamical equation (172) is not the topic of this paper, but we refer to [40] where the general way of incorporation of special representations into quantum field functional theory was outlined.

8. Conclusions

Our treatment shows that nonabelian gauge theories are considerably more complicated than abelian gauge theories. Hence for nonabelian gauge theories an additional discussion is needed about the results obtained in the preceding sections. Like in quantum

electrodynamics, also in the case of quantum chromodynamics we succeeded in deriving a functional energy equation (169) which allows a selfconsistent calculation of functional energy eigenstates depending only on the transversal branches of gauge field variables. Therefore, within the framework of algebraic quantum chromodynamics in temporal gauge it is possible to isolate a selfconsistent subsystem from the whole set of dynamical equations, of which the variables correspond to the Coulomb gauge variables. In quantum electrodynamics we were able to show the equivalence of this system to the algebraic equations in Coulomb gauge. As an exact classical evaluation of the Coulomb gauge Hamiltonian does not exist in chromodynamics, a derivation of the corresponding functional energy equations cannot be performed and thus we cannot prove (169) to be the algebraic Coulomb gauge version of QCD. Nevertheless, due to the complete analogy of the derivation of isolated subsystems in both QED and QCD, we postulate that we have directly obtained the quantum version of the Coulomb gauge in chromodynamics. One characteristic feature of Coulomb gauge in QED is the absence of constraints. This also holds for the system (169). With respect to the invariance conditions of physical states against gauge transformations, one therefore has to investigate whether such an invariance is guaranteed for the solutions of (169). In spite of the difficulties mentioned above, the Coulomb gauge is a frequently investigated gauge in QCD, for reviews see [2], [3]. The Coulomb gauge is defined by $\nabla \cdot \mathbf{A}^a = 0$. With respect to local gauge transformation the Coulomb gauge in QCD means a complete gauge fixing, i.e. in Coulomb gauge no residual local gauge transformations remain (in contrast to QED). This can be seen by forming the divergence of an infinitesimal gauge transformation for \mathbf{A}_a :

$$\mathbf{A}_a(\mathbf{r}, t)' = \mathbf{A}_a(\mathbf{r}, t) + \frac{1}{g} \nabla \omega_a(\mathbf{r}) + f_{abc} \omega_b(\mathbf{r}) \mathbf{A}_c(\mathbf{r}, t). \tag{173}$$

One obtains for $\Delta\omega(\mathbf{r}) = 0$ (which is the gauge condition of QED)

$$\nabla \cdot \mathbf{A}_a(\mathbf{r}, t)' = f_{abc} \mathbf{A}_c(\mathbf{r}, t) \cdot \nabla \omega_b(\mathbf{r}) \tag{174}$$

which for $\omega_a \neq \epsilon_a$ is unequal zero. Hence any local gauge transformation destroys the gauge condition. Therefore, in Coulomb gauge no constraints resulting from local residual gauge transformations have to be imposed on the states. The only remaining gauge

transformations are global ones with $\omega_a(\mathbf{r}) = \epsilon_a$, which leave the Coulomb gauge invariant. Thus the invariance against global gauge transformations is the only condition which has to be satisfied by the eigenstates of (169). But this invariance condition can be directly incorporated into the set of eigensolutions of (169) by construction of irreducible one-dimensional representations of the global gauge group within this set. The solution of this problem is standard and thus will not be discussed here. Furthermore, the invariance of the energy operator of (169) against global gauge transformation has to be proved in detail, but the success is secured since the Coulomb terms result from the global gauge invariant expression of the longitudinal color electric field energy.

Summarizing our discussion we have obtained a functional energy equation (169), which fulfills all requirements of an algebraic version of quantum chromodynamics in Coulomb gauge and which allows a selfconsistent calculation of functional algebraic energy eigenstates. The residual gauge invariance of states in this gauge is reduced to invariance against global gauge transformations, which can be guaranteed by standard methods of group theory.

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